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Expansions of quadratic numbers in a p-adic continued fraction

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The theory of real continued fractions plays a central role in real Diophantine Approximation for many different reasons, in particular because the convergents of the simple continued fraction expansion of a real number α give the best rational approximations to α. Motivated by the same type of questions, several authors (Mahler, Schneider, Ruban, Bundschuch and Browkin) have generalized the theory of real continued fractions to the ℓ-adic case in various ways.

The theory of ℓ-adic continued fractions presents many differences with respect to the real case. First of all, there is no canonical way to define a continued fraction expansion in this context, as the expansion depends on the chosen system of residues mod ℓ, and the basic properties of finiteness and periodicity change with this choice. The ℓ-adic process which is the most similar to the classical real one was mentioned for the first time in one of the earliest papers on the subject by Mahler and then studied accurately by Ruban, who showed that these continued fractions enjoy nice ergodic properties.

In a joint work with F. Veneziano and U. Zannier we investigate questions about finiteness and periodicity of Ruban's continued fraction expansions.

In the classical real case, a real number has finite continued fraction expansion if and only if the number is rational, and Lagrange's theorem ensures that a real number has an infinite periodic continued fraction expansion if and only if it is quadratic irrational.

For Ruban's continued fraction expansion instead, also rational numbers can have periodic continued fractions, as showed by Laohakosol and independently by Wang. Moreover, for quadratic irrationals, no full analogue of Lagrange's theorem holds, as showed by Ooto, but it was not known how to decide whether the expansion for a given quadratic number is or is not periodic. In our work, we give a completely general algorithm in this sense which, somewhat surprisingly, depends on the "real" values of the complete quotients appearing in the ℓ-adic continued fraction expansion:

Theorem 1 *Let $\alpha \in \mathbb{Q}_\ell \setminus \mathbb{Q}$ be a quadratic irrational over $\mathbb{Q}$. Then, the Ruban continued fraction expansion of α is periodic if and only if there exists a unique real embedding $j : \mathbb{Q}(\alpha) \to \mathbb{R}$ such that the image of each complete quotient α_n under the map j is positive.*

Moreover, there is an effective constant N_α with the property that, either $\exists\, n \leq N_\alpha$ such that α_n does not have a positive real embedding, and therefore the expansion is not periodic, or $\exists\, n_1 < n_2 \leq N_\alpha$ such that $\alpha_{n_1} = \alpha_{n_2}$, hence the expansion is periodic.

If $\alpha \in \mathbb{Q}_\ell$ is of the form $\alpha = \frac{b+\sqrt{\Delta}}{c}$ with b, c, Δ integers and $\Delta > 0$ not a square in $\mathbb{Q}_\ell$, then the constant N_α in the Theorem can be taken equal to $bc + 2(c\sqrt{\Delta} + 1)^3$.

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