COMPLEX ELLIPTIC CURVES

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Universiteit Leiden
2023

Date of this version: January 19, 2023

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1. Elliptic integrals

Although elliptic curves are algebraic curves, their name still shows their origin in the differential and integral calculus that was developed in the 17th and 18th century, and that became the main (and not so algebraic) part of what is nowadays a 'basic mathematical education'. This note shows how the complex analytic point of view yields a blueprint for the now more common algebraic approach starting form 'algebraic curves defined by a Weierstrass equation $y^2 = x^3 - ax - b$'.

In calculus, one tries to integrate the *differentials* $f(t)dt$ associated with, say, a real-valued function f on the real line. As is well known, such integrals are related to the area of certain surfaces bounded by the graph of f. Explicit integration of the differential $f(t)dt$, which amounts to finding an anti-derivative F satisfying $dF/dt = f$, can only be performed for a few elementary differentials. These include polynomial differentials $t^k dt$ with $k \in \mathbf{Z}_{\geq 0}$, rational differentials $(t-\alpha)^{-k}$ with $k \in \mathbf{Z}_{>0}$, and a few 'exponential differentials' such as $e^t dt$ and $\sin t\, dt$. Over the complex numbers, any rational differential can be written as a sum of such differentials.

Exercise 1. Show that every rational function $f \in \mathbf{C}(t)$ can be written as unique $\mathbf{C}$-linear combination of monomials t^k with $k \in \mathbf{Z}_{\geq 0}$ and fractions $(t-\alpha)^{-k}$ with $\alpha \in \mathbf{C}$ and $k \in \mathbf{Z}_{\geq 1}$. Use this representation to write $\int f(t)dt$ as a sum of elementary functions. [Hint: partial fraction expansion.]

Even if one restricts to polynomial or rational functions f, already the problem of computing the *length* of the graph of f, an old problem known as the 'rectification' of plane curves, leads to the non-elementary differential $\sqrt{1+f'(t)^2}dt$. If $R \in \mathbf{C}(x,y)$ is a rational function and $f \in \mathbf{C}[t]$ a polynomial that is not a square, the differential $R(t, \sqrt{f(t)})dt$ is called *hyperelliptic*. We can and will always suppose that f is *separable*, i.e., it has no multiple roots. If f is of degree 1, one can transform $R(t, \sqrt{f(t)})dt$ into a rational differential by taking $\sqrt{f(t)}$ as a new variable. If f is quadratic, one can apply a linear transformation $t \mapsto at+b$ to reduce to the case $f(t) = 1-t^2$. We will see in a moment that the resulting integrals are closely related to the problem of computing lengths of circular arcs or, what amounts to the same thing, inverting trigonometric functions. If f is of degree 3 or 4 and squarefree, the differential $R(t, \sqrt{f(t)})dt$ is said to be *elliptic*, as it arises in the computation of arc lengths of ellipses.

Exercise 2. Show that, for $c \neq 0$, the length of the ellipse with equation $y^2 = c^2(1-x^2)$ in $\mathbf{R}^2$ equals

$$2\int_{-1}^{1} \sqrt{\frac{1+(c^2-1)t^2}{1-t^2}}\,dt = 2\int_{-\pi/2}^{\pi/2} \sqrt{1+(c^2-1)\sin^2\phi}\,d\phi,$$

and that the differential $\sqrt{\frac{1+(c^2-1)t^2}{1-t^2}}\,dt$ is elliptic for $c^2 \neq 1$.

Elliptic differentials lead naturally to the study of elliptic functions and elliptic curves. In a similar way, the case of f of higher degree gives rise to hyperelliptic curves. More generally, it has gradually become clear during the 19th century that an algebraic differential $R(t,u)dt$, with R a rational function and t and u satisfying some polynomial relation

$P(t, u) = 0$, should be studied as an object living on the plane algebraic curve defined by the equation $P(x, y) = 0$. For hyperelliptic differentials, this is the hyperelliptic curve given by the equation $y^2 = f(x)$.

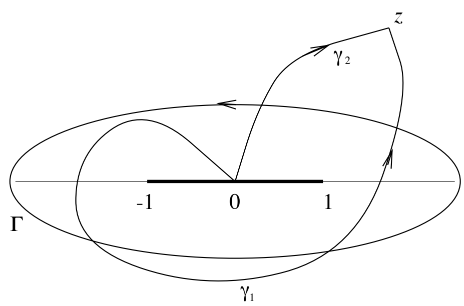
As an instructive example, we consider the differential $\omega = dt/\sqrt{1-t^2}$ related to the arc length of the unit circle. The reader can easily check that the graph of the function $f(t) = \sqrt{1-t^2}$ on the real interval $[-1, 1]$ is a semicircle, and that we have $\omega = \sqrt{1 + f'(t)^2}dt$. We attempt to define a map

$$\phi : z \longmapsto \int_0^z \omega = \int_0^z \frac{dt}{\sqrt{1-t^2}} \tag{1.1}$$

as a function on $\mathbf{C}$. Note that ω has integrable singularities at the points $t = \pm 1$.

There are two problems with the map ϕ. First of all, there is no canonical definition of a square root $\sqrt{1-t^2}$ for $t \in \mathbf{C}$. One can select a specific square root for $t \in [-1, 1]$ or t on the imaginary axis, when $1 - t^2$ is real and positive, but such extensions do not yield an obvious choice for, say, $t = \pm 2$. A rather uncanonical way out is the possibility of making a *branch cut*. This means that one defines ϕ not on $\mathbf{C}$, but on a subset of $\mathbf{C}$, such as $\mathbf{C} \setminus [-1, 1]$, on which $\sqrt{1-t^2}$ admits a single-valued branch.

If one makes the proposed branch cut and chooses a branch of ω, a second problem arises: two different paths of integration can give rise to different values of $\phi(z)$, so the map ϕ is not well-defined.



The difference between the two values of $\phi(z)$ for the paths γ_1 and γ_2 in the picture is the value of the integral $\oint \omega$ along a simple closed curve Γ around the two singular points $t = \pm 1$ of ω. One can compute this contour integral in various ways.

Exercise 3. Apply the residue theorem to evaluate $\oint_\Gamma \omega$. [Answer: $\pm 2\pi$.]

As the value of the real integral $\int_{-1}^1 \omega$ is the length of a semicircle of radius 1, one easily sees that $\oint \omega$ has value $\pm 2\pi$, with the sign depending on the choice of the square root $\sqrt{1-t^2}$ along the path of integration. From the topology of $\mathbf{C} \setminus [-1, 1]$, it is clear that the values of $\phi(z)$ computed along different paths always differ by a multiple of 2π.

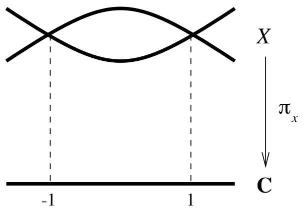
There is a canonical reparation of the definition of ϕ that makes ϕ into a well-defined map on a 'natural domain' for ω. Rather than defining ϕ on $\mathbf{C}$ minus some branch cut, one considers the set

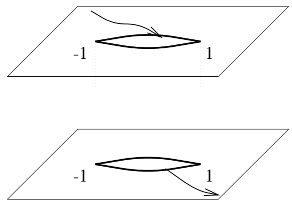
$$X = \{(x, y) \in \mathbf{C}^2 : y^2 = 1 - x^2\}.$$

This set comes with a natural projection $\pi_x : X \to \mathbf{C}$ defined by $(x, y) \mapsto x$. Given any point $t \in \mathbf{C}$, the *fiber* $\pi_x^{-1}(t)$ consists of the points $(t, u) \in \mathbf{C}^2$ for which u is a square root of $1 - t^2$. For $t \neq \pm 1$, there are exactly 2 such points, and one says that the projection $\pi_x : X \to \mathbf{C}$ is *generically* 2-to-1. For the *branch points* $t = \pm 1$ there is only one point in the fiber.

As the complex curve X is a subset of $\mathbf{C}^2$, one cannot immediately picture X. There are two approximate solutions. The first consists of drawing $\mathbf{C}$ as a 1-dimensional object and representing π_X as in the picture below. One disadvantage of this method is that the points $(\pm 1, 0)$ on X appear to be of a special nature. The symmetry in x and y in the definition of X shows that this cannot be the case.

Exercise 4. Draw the corresponding picture for the map $\pi_y : X \to \mathbf{C}$ sending (x, y) to y. Where are the points $(\pm 1, 0)$ in this picture?





Another, usually somewhat more complicated way to visualize X is to take two copies of $\mathbf{C}$ and 'glue them along a branch cut' as suggested in the picture. In the space obtained, paths passing through the branch cut in one copy of $\mathbf{C}$ emerge on the 'opposite side' of the branch cut in the *other* copy. A moment's reflection shows that, topologically, the resulting surface is homeomorphic to a cylinder. The path Γ becomes the simplest incontractible path on X. It is immediate from the picture that every path $0 \to z$ in $\mathbf{C}$ that does not pass through the branch points ± 1 can uniquely be lifted to a path $x_0 = (0, 1) \to (z, w)$, where w is a square root of $1 - z^2$ that is determined by the path $0 \to z$. The function $t \to \sqrt{1 - t^2}$, which has no natural definition on $\mathbf{C}$, has by construction a natural definition on X: it is the function $(t, u) \to u$. It is now also clear how one should integrate the differential dt/u, which we denote again by ω, along any path in X. We arrive at a definition of ϕ on X rather than $\mathbf{C}$, which is given by

$$\phi(x) = \int_{x_0}^{x} \omega = \int_{x_0}^{x} \frac{dt}{u} \qquad \text{for } x \in X \subset \mathbf{C}^2.$$

The integral is taken along X, and as we have a choice of paths its value is only determined up to multiples of 2π. This means that $\phi(z)$ is well defined as an element of the factor group $\mathbf{C}/2\pi\mathbf{Z}$ of $\mathbf{C}$. The elements of this group can be viewed as the complex numbers in the infinite strip $\{z : -\pi \leq \mathrm{Re}(z) \leq \pi\}$, where for any $r \in \mathbf{R}$, the elements $-\pi + ir$ and $\pi + ir$ on the boundary are identified. Topologically, one notes that just like X, the group $\mathbf{C}/2\pi\mathbf{Z}$ is a cylinder. The following theorem is therefore not so surprising.

1.2. Theorem. *The integration of the differential ω induces a bijection $\phi : X \xrightarrow{\sim} \mathbf{C}/2\pi\mathbf{Z}$.*

We leave it to the reader to give a complete proof of the theorem, as indicated in the exercises, and to show that ϕ is in a natural way a homeomorphism of topological spaces.

Theorem 1.2 has a number of interesting consequences. It shows that the set X, which is the *algebraic curve* in $\mathbf{C}^2$ defined by the equation $x^2+y^2=1$, is in a natural way a group. From the map ϕ, which is defined by means of an integral, it is not immediately clear what the sum of two points on X should be. However, in this case we know from calculus that integration of the real differential $\omega = dt/\sqrt{1-t^2}$ yields the function $\arcsin t$, a somewhat artificially constructed inverse to the sine function. In fact, our carefully constructed map ϕ has an *inverse* which is much easier to handle. From the observation that $\pi_x \circ \phi^{-1}$ is in fact the sine function and the identity $\phi^{-1}(0) = (0, 1)$, the following theorem is now immediate.

1.3. Theorem. *The inverse $\phi^{-1} : \mathbf{C}/2\pi\mathbf{Z} \to X$ of the bijection ϕ in* 1.2 *is given by $\phi^{-1}(z) = (\sin z, \cos z)$.*

It follows from 1.3 that we may describe the natural addition on X by the formula $(\sin\alpha, \cos\alpha) + (\sin\beta, \cos\beta) = (\sin(\alpha+\beta), \cos(\alpha+\beta))$. From the addition formulas for the sine and cosine functions one deduces that the group law on X is in fact given by the simple polynomial formula

$$(x_1, y_1) + (x_2, y_2) = (x_1y_2 + x_2y_1, y_1y_2 - x_1x_2). \tag{1.4}$$

The unit element of X is the point $(0, 1)$, and the inverse of $(x, y) \in X$ is the point $(-x, y)$. This shows that X is in fact an *algebraic group*: for every subfield of $K \subset \mathbf{C}$, such as $\mathbf{Q}$ or $\mathbf{Q}(i)$, the set $X(K) \subset K^2$ of K-valued points of X is an abelian group. A picture of the real locus $X(\mathbf{R}) = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ explains why X is known as the *circle group*.

Exercise 5. Draw a picture of $X(\mathbf{R})$ and give a geometric description of the group law.

As we have constructed the circle group by analytic means, via the construction of ϕ, it is not immediately obvious that formula 1.4 defines a group structure on $X(K)$ for arbitrary fields K. Clearly, there is no 'analytical parametrization' ϕ^{-1} of X if we replace $\mathbf{C}$ by a field of positive characteristic, such as the finite field $\mathbf{F}_p$. Therefore, the following theorem does require a proof.

1.5. Theorem. *Let K be a field. Then formula* 1.4 *defines a group structure on the set $X(K) = \{(x, y) \in K^2 : x^2 + y^2 = 1\}$.*

Proof. It is straightforward but unenlightening to check the group axioms from the definition. One can however observe that under the injective map $X(K) \to \mathrm{SL}_2(K)$ given by $(x, y) \mapsto \left(\begin{smallmatrix} y & -x \\ x & y \end{smallmatrix}\right)$, the operation given by 1.4 corresponds to the well known matrix multiplication. It follows that 1.4 defines a group structure on $X(K)$. $\square$

Exercise 6. Let K be a field of characteristic 2. Show that the projection $\pi_x : X(K) \to K$ mapping (x, y) to x is a group isomorphism.

As is shown by the preceding exercise, one has to be careful when interpreting pictures over the complex numbers—such as that of the generically 2-to-1 projection $\pi_x : X(\mathbf{C}) \to \mathbf{C}$ above—in positive characteristic.
We now replace the differential $dt/\sqrt{1-t^2}$ in the preceding example by an elliptic differential $dt/\sqrt{f(t)}$ for some squarefree polynomial f of degree 3 or 4. We will see that the complex 'unit circle' $X = \{(x,y) : x^2 + y^2 = 1\}$ gets replaced by the *elliptic curve* $E = \{(x,y) : y^2 = f(x)\}$, and the map $\phi^{-1} : z \to (\sin z, \cos z)$ by a map $z \mapsto (P(z), P'(z))$ for some *elliptic function* P. As in the case of the circle, the analytic parametrization by elliptic functions will equip E with a group structure. In the next section, we will give a geometric description of the group law and derive explicit algebraic addition formulas.

For a quadratic polynomial f a simple transformation $t \mapsto at + b$ suffices to map the roots of f to ± 1, yielding a differential with $f(t) = 1 - t^2$. In the elliptic case, the situation is more complicated. One can apply *Möbius transformations* $t \mapsto \frac{at+b}{ct+d}$ with $ad - bc \neq 0$, which act bijectively on the compactified complex plane $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$, commonly referred to as the *Riemann sphere*.

1.6. Lemma. *Under a Möbius transformation $t \mapsto \frac{at+b}{ct+d}$, elliptic differentials transform into elliptic differentials.*

Proof. It suffices to check this for a differential $\omega = dt/\sqrt{f(t)}$, with $f(t) = \sum_{k=0}^4 r_k t^k$ of degree 3 or 4. One finds that ω is transformed into

$$\omega^* = \frac{1}{\sqrt{f(\frac{at+b}{ct+d})}} d(\frac{at+b}{ct+d}) = \frac{(ad-bc)\ dt}{\sqrt{\sum_{k=0}^4 r_k (at+b)^k (ct+d)^{4-k}}}.$$

The polynomial $g(t) = \sum_{k=0}^4 r_k (at+b)^k (ct+d)^{4-k}$ is of degree at most 4. We leave it to the reader to verify that the degree is at least 3, so that ω^* is again elliptic. □

Exercise 7. Show that if the polynomial f in the preceding proof is of degree 4, the transformed differential has a polynomial g of degree 3 if and only if the Möbius transformation maps ∞ to a zero of f.

Möbius transformations can be used to map three of the roots of f to prescribed values in $\mathbf{P}^1(\mathbf{C})$. Different choices lead to different *normal forms* for elliptic differentials.

Exercise 8. Show that every elliptic differential $R(t, \sqrt{f(t)})$ can be transformed by a Möbius transformation into a differential for which f has one the following shapes:

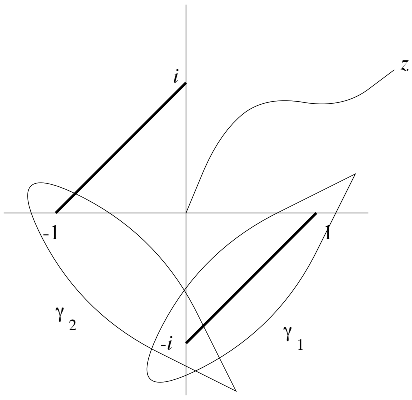
$$f(t) = t(t-1)(t-\lambda) \qquad f(t) = t^3 + at + b \qquad f(t) = (1-t^2)(1-k^2t^2).$$

[The corresponding normal forms are named after Legendre, Weierstrass and Jacobi.]

As an example of an elliptic differential, we consider the differential $\omega = dt/\sqrt{1-t^4}$ related to the rectification of the *lemniscate*. In order to find the analogue of 1.2 for ω, we start as in 1.1 and try to define a map

$$\psi : z \longmapsto \int_0^z \omega = \int_0^z \frac{dt}{\sqrt{1-t^4}}.$$

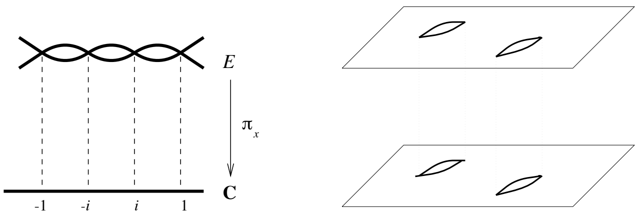
This time ω has integrable singularities in the 4th roots of unity, and it becomes single-valued if we make make branch cuts $[-1, i]$ and $[-i, 1]$. The picture in the complex plane is as follows.



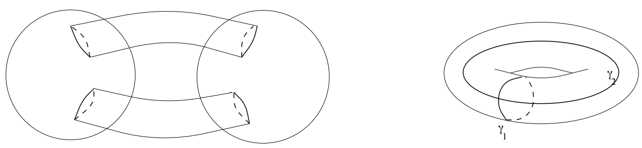
In order to obtain a natural domain for ω, we consider the algebraic curve E in $\mathbf{C}^2$ with equation

$$E = \{(x, y) : y^2 = 1 - x^4\} \subset \mathbf{C}^2.$$

As in our previous example, the projection $\pi_x : E \to \mathbf{C}$ on the x-coordinate is generically 2-to-1 with branch points at ± 1 and $\pm i$. A topological model for E can be obtained by glueing two copies of $\mathbf{C}$ along our two branch cuts.



As $\psi(z)$ converges for $z \to \infty$, it makes sense to view ψ as a map on the Riemann sphere $\mathbf{P}^1(\mathbf{C})$. This means that we have to modify the picture above and add two 'points at infinity' to E, one coming from each copy of $\mathbf{C}$ in our topological picture. We write E again for the completed curve. We see from the picture that the glueing of two spheres along two branch cuts yields a doughnut-shaped surface known as a *torus*. On this surface, there are *two* independent incontractible paths. Under π_x, they are mapped to the paths γ_1 and γ_2 in our earlier picture. One can show that the homotopy classes of these paths generate the fundamental group $\pi(E) = \mathbf{Z} \times \mathbf{Z}$ of E.



Exercise 9. Show that $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ is a contractible path on E.

It follows that the values of ψ are uniquely determined as elements of the factor group $\mathbf{C}/(\mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2)$, where the *periods* λ_1 and λ_2 are defined as $\lambda_i = \oint_{\gamma_i} \omega$ for $i = 1, 2$. From our initial picture we see that the path γ_1 maps to γ_2 onder multiplication by $-i$. As $1-t^4$ is invariant under this transformation, we deduce that we have $\lambda_2 = -i\lambda_1$. The subgroup $\Lambda = \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2$ is a rectangular *lattice* in $\mathbf{C}$, and the factor group $\mathbf{C}/\Lambda$ is therefore topologically a torus. We have the following analogue of 1.2.

1.7. Theorem. *The integration of the differential* $\omega = \frac{dx}{y}$ *along the completed curve* E *induces a bijection* $\psi : E \xrightarrow{\sim} \mathbf{C}/(\mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2)$.

For a complete proof of 1.7, and for similar results for other elliptic differentials, we refer to the exercises.

As a consequence of 1.7, we see that the *elliptic curve* E carries a natural group structure. Let the inverse function $\psi^{-1} : \mathbf{C}/(\mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2) \xrightarrow{\sim} E$ be given by $\psi^{-1}(z) = (P(z), Q(z))$. As the derivative of ψ in (x, y) with respect to x is by construction equal to $1/y$, the derivative of P in $z = \psi((x, y))$ equals $y = Q(z)$. We conclude that as in the previous example, the inverse of ψ is of the form $\psi^{-1}(z) = (P(z), P'(z))$ for some *elliptic function* P. As E has two points at infinity, the 'lemniscatic P-function' $P(z)$ has a pole in two values of z in $\mathbf{C}/(\mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2)$. At all other points, it is holomorphic. From the equation of E, it is clear that P is a solution to the differential equation

$$(P')^2 = 1 - P^4.$$

As a function on $\mathbf{C}$, it has even stronger periodicity properties than the sine function: it is *double-periodic* with independent periods λ_1 and λ_2.

Exercise 10. Define $p = \int_{-1}^{1} dt/\sqrt{1-t^4} \approx 2.622057556$, the elliptic analogue of $\pi = \int_{-1}^{1} dt/\sqrt{1-t^2}$. Show that we can take $\lambda_1 = p + ip$ and $\lambda_2 = p - ip$ in 1.7, and that the elliptic function P has poles in $\lambda_1/2$ and $\lambda_2/2$. Are these poles simple?

Just as the sine and cosine functions are more convenient to handle than the arcsine and arccosine functions arising from the integration of $dt/\sqrt{1-t^2}$, the functions P and P' constructed above are easier to study than the function $\psi(z) = \int^z dt/\sqrt{1-t^4}$. By clever substitutions in the integral defining ψ, one can prove Fagnano's duplication formula

$$P(2z) = \frac{2P(z)P'(z)}{1 + P(z)^4},$$

which dates back to 1718. Euler extended this result in 1752 and found the general addition formula

$$P(z_1) + P(z_2) = \frac{P(z_1)P'(z_2) + P'(z_1)P(z_2)}{1 + P(z_1)^2 P(z_2)^2}. \tag{1.8}$$

for the lemniscatic P-function.

The next section is devoted to the analysis of analytic functions on an arbitrary torus. We will show directly that *all* tori come with functions satisfying algebraic addition formulas.

Exercises.

11. Adapt the statement in exercise 1 for rational functions f with real coefficients and show that $\int f(t)dt$ can be expressed in terms of 'real elementary functions'.

12. Show that the map ϕ in 1.1 is well-defined as a map on the complex upper half plane $\mathcal{H} = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$, provided that we fix a branch of $\sqrt{1-t^2}$ on $\mathcal{H}$. Show that for the branch that is positive on $i\mathbf{R}_{>0}$, we obtain a bijective map $\phi : \mathcal{H} \to S$ to the semi-infinite strip $S = \{z \in \mathbf{C} : \mathrm{Im} z > 0 \text{ and } -\pi/2 < \mathrm{Re} z < \pi/2\}$. Derive theorem 1.2 from this statement.
[Hint: determine the image of the real axis under ϕ.]

*13. Show that the map ϕ in 1.2 is an isomorphism of complex analytic spaces, i.e., a biholomorphic map between open Riemann surfaces.

14. A *lemniscate of Bernoulli* is the set L of points X in the Euclidean plane for which the product of the distances XP_1 and XP_2, with P_1 and P_2 given points at distance $P_1P_2 = 2d > 0$, is equal to d^2.
    a. Show that for a suitable choice of coordinates, the equation for L is $(x^2+y^2)^2 = x^2 - y^2$ or, in polar coordinates, $r^2 = \cos 2\phi$. Sketch this curve.
    b. Show that the arclength of the 'unit lemniscate' in (a) equals $2p$, with p defined as in exercise 10. [Note the similarity with the arclength of the unit circle, which equals 2π.]

15. This exercise gives a 'proof by algebraic manipulation' of Fagnano's duplication formula for the lemniscatic P-function.
    a. Show that the substitution $t = 2v/(1+v^2)$ transforms the differential $dt/\sqrt{1-t^2}$ to the rational differential $2dv/(1+v^2)$.
    b. Show that the substitution $t^2 = 2v^2/(1+v^4)$ transforms the differential $dt/\sqrt{1-t^4}$ to the differential $\sqrt{2}dv/\sqrt{1+v^4}$, and that the subsequent substitution $v^2 = 2w^2/(1-w^4)$ leads to the differential $2dw/\sqrt{1-w^4}$.
    c. Derive the relation $t = 2w\sqrt{1-w^4}/(1+w^4)$ for variables in (b), and use this to prove Fagnano's formula.

16. On the complex upper half plane $\mathcal{H}$, we can uniquely define a function

$$\phi(z) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(4-t^2)}}$$

by integrating along paths in $\mathcal{H}$. We use the branch of $\sqrt{(1-t^2)(4-t^2)}$ that is positive on $i\mathbf{R}_{>0}$. Define $A, B \in \mathbf{C}$ by $A = \lim_{z\to 1} \phi(z)$ and $A + B = \lim_{z\to 2} \phi(z)$.
    a. Show that A is real and B purely imaginary, and that we have $\lim_{z\to\infty} \phi(z) = B$.
    b. Show that the map ϕ extends to a bijection between the completion of the elliptic curve $y^2 = (1-x^2)(4-x^2)$ and the torus $\mathbf{C}/\Lambda$ with $\Lambda = \mathbf{Z}\cdot 4A + \mathbf{Z}\cdot 2B$.

17. Prove theorem 1.7. [Hint: imitate the previous exercise.]

*18. Show that the map ψ in 1.7 is an isomorphism of complex analytic spaces, i.e., a biholomorphic map between compact Riemann surfaces.

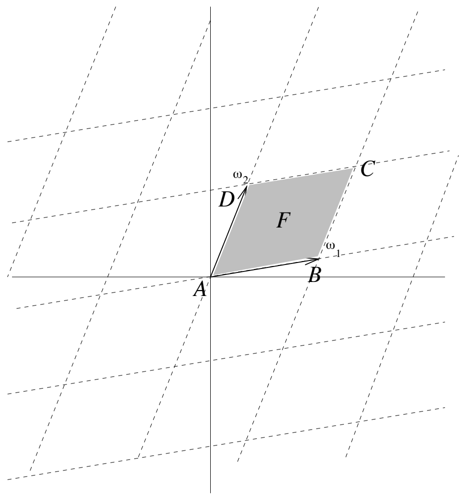
2. Elliptic functions

In this section, we will develop the basic theory of double-periodic functions encountered in the previous section.

A *lattice* in $\mathbf{C}$ is a discrete subgroup of $\mathbf{C}$ of rank 2. It has the form $\Lambda = \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2$ for some $\mathbf{R}$-basis $\{\lambda_1, \lambda_2\}$ of $\mathbf{C}$. One often writes $\Lambda = [\lambda_1, \lambda_2]$. The factor group $T = \mathbf{C}/\Lambda$ is called a *complex torus*. A *fundamental domain* for T is a connected subset $F \subset \mathbf{C}$ for which every $z \in \mathbf{C}$ can uniquely be written as $z = f + \lambda$ with $f \in F$ and $\lambda \in \Lambda$. Note that any translate of a fundamental domain is again a fundamental domain. For every choice $\{\lambda_1, \lambda_2\}$ of a $\mathbf{Z}$-basis of Λ, the set

$$F = \{r_1\lambda_1 + r_2\lambda_2 : r_1, r_2 \in \mathbf{R}, \quad 0 \le r_1, r_2 < 1\} \subset \mathbf{C}$$

is a fundamental domain for T.



An *elliptic function* with respect to Λ is a meromorphic function f on $\mathbf{C}$ that satisfies $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$. Such a function is uniquely determined by its values on a fundamental domain. An elliptic function factors as $f : \mathbf{C} \to \mathbf{C}/\Lambda = T \to \mathbf{P}^1(\mathbf{C})$, so we can identify the set of elliptic functions with respect to Λ with the set $\mathcal{M}(T)$ of meromorphic functions on $T = \mathbf{C}/\Lambda$. Sums and quotients of meromorphic functions are again meromorphic, so the set $\mathcal{M}(T)$ is actually a field, the *elliptic function field* corresponding to T.

As T is compact, any holomorphic function $f \in \mathcal{M}(T)$ is bounded on T. This means that f comes from a bounded holomorphic function on $\mathbf{C}$, so by Liouville's theorem f is constant. We conclude that any non-constant elliptic function has at least one pole.

Exercise 1. Show that for any non-constant $f \in \mathcal{M}(T)$, the map $f : T \to \mathbf{P}^1(\mathbf{C})$ is surjective.

The most convenient way to describe the zeroes and poles of a function $f \in \mathcal{M}(T)$ is to define its associated *divisor*. The *divisor group* $\mathrm{Div}(T)$ is the free abelian group generated by the points of T. Equivalently, a divisor

$$D = \sum_{w \in T} n_w[w] \in \mathrm{Div}(T) = \bigoplus_{w \in T} \mathbf{Z}$$

is a *finite* formal sum of points of T with integer coefficients.

The divisor group $\mathrm{Div}(T)$ comes with canonical surjective homomorphisms to T and $\mathbf{Z}$. The *summation map* $\Sigma : \mathrm{Div}(T) \to T$ sends $\sum_{w\in T} n_w[w]$ to $\sum_{w \in T} n_w w$. The *degree map* $\deg : \mathrm{Div}(T) \to \mathbf{Z}$ sends $\sum_{w\in T} n_w[w]$ to $\sum_{w\in T} n_w$. The kernel of the degree map is the subgroup $\mathrm{Div}^0(T) \subset \mathrm{Div}(T)$ of divisors of degree zero.

The *order* $\mathrm{ord}_w(f) \in \mathbf{Z}$ of a non-zero function $f \in \mathcal{M}(T)^*$ at a point $w \in T$ is the minimum of all k for which the coefficient c_k in the Laurent expansion $f(z) = \sum_k c_k(z-w)^k$ of f around w is non-zero. If we view poles as zeroes of negative order, $\mathrm{ord}_w(f) \in \mathbf{Z}$ is simply the order of the zero of f in w.

A meromorphic function $f \in \mathcal{M}(T)^*$ has only finitely many zeroes and poles on the compact torus T, so the divisor map

$$\begin{aligned} \mathrm{div} : \mathcal{M}(T)^* &\longrightarrow \mathrm{Div}(T) \\ f &\longmapsto (f) = \sum_{w \in T} \mathrm{ord}_w(f)[w] \end{aligned}$$

is a well-defined homomorphism. The divisors in $\mathrm{Div}(T)$ coming from elliptic functions are called *principal divisors*. We will prove that a divisor is principal if and only if it is in the kernel of both Σ and deg.

2.1. Theorem. *Let $T = \mathbf{C}/\Lambda$ be a torus. Then there is an exact sequence of abelian groups*

$$1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{M}(T)^* \xrightarrow{\mathrm{div}} \mathrm{Div}^0(T) \xrightarrow{\Sigma} T \longrightarrow 1.$$

As only constant functions on T are without zeroes and poles, the sequence is exact at $\mathcal{M}(T)^*$. The proof of the exactness at $\mathrm{Div}^0(T)$ consists of two parts. We first prove that principal divisors are of degree zero and in the kernel of the summation map. These are exactly the statements (ii) and (iii) of the lemma below.

2.2. Lemma. *Let f be a non-zero elliptic function on T. Then the following holds.*

(i) $\sum_{w\in T} \mathrm{res}_w(f) = 0$.
(ii) $\sum_{w\in T} \mathrm{ord}_w(f) = 0$.
(iii) $\sum_{w\in T} \mathrm{ord}_w(f) \cdot w = 0 \in T$.

Proof. Let F be a fundamental domain for T, and suppose—after translating F when necessary—that none of the zeroes and poles of f lies on the boundary ∂F of F. Then the expressions of the lemma are the values of the contour integrals

$$\frac{1}{2\pi i}\oint_{\partial F} f(z)dz, \qquad \frac{1}{2\pi i}\oint_{\partial F} \frac{f'(z)}{f(z)}dz, \qquad \frac{1}{2\pi i}\oint_{\partial F} z\frac{f'(z)}{f(z)}dz.$$

The first two integrals vanish since, by the periodicity of f and f'/f, the integrals along opposite sides of the parallellogram F coincide; as these sides are traversed in opposite directions, their contributions to the integral cancel.

The function $z\frac{f'(z)}{f(z)}$ is not periodic, but we can still compute the contribution to the integral coming from opposite sides AB and $DC = \{z + \lambda_2 : z \in AB\}$ of F, as indicated in the earlier picture. We find

$$\int_{AB} z\frac{f'(z)}{f(z)}dz + \int_{CD} z\frac{f'(z)}{f(z)}dz = \int_{AB} z\frac{f'(z)}{f(z)}dz - \int_{AB}(z+\lambda_2)\frac{f'(z)}{f(z)}dz = -\lambda_2 \int_{AB} \frac{f'(z)}{f(z)}dz.$$

As the integral $\frac{1}{2\pi i}\int_{AB}\frac{f'(z)}{f(z)}dz$ is the winding number of the *closed* path described by $f(z)$ if z ranges from A to B along ∂F, $\frac{1}{2\pi i}$ times the value of the displayed integral is an integral multiple of λ_2, hence in Λ. The same holds for the other half $\int_{BC} + \int_{DA}$ of the integral, which yields an integral multiple of λ_1. The complete integral now assumes a value in Λ, and (iii) follows.

Assertion (ii) of the lemma shows that an elliptic function has as many zeroes as it has poles on T, if we count multiplicities. The number of zeroes (or, equivalently, poles) of an elliptic function f, counted with multiplicity, is called the *order* of f. Equivalently, it is the degree of the *polar divisor* $\sum_w \max(0, -\mathrm{ord}_w(f)) \cdot (w)$ of f. It follows from (i) that the order of an elliptic function cannot be equal to 1.

Exercise 2. Define the order of a meromorphic function on $\mathbf{P}^1(\mathbf{C})$, and show that functions of arbitrary order exist.

In order to complete the proof of 2.1, we need to show that a divisor of degree zero that is in the kernel of Σ actually corresponds to a function on T. This means that we somehow have to construct these functions.

Function theory provides us with two methods to construct meromorphic functions with prescribed zeroes or poles. An additive method consists in writing down a series expansion for the 'simplest elliptic function' associated to the lattice Λ, the *Weierstrass-$\wp$-function* $\wp_\Lambda(z)$. This is an even function of order 2 on T, which has a double pole at $0 \in T$. It is given by

$$\wp(z) = \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right). \tag{2.3}$$

In order to show that the defining series converges uniformly on compact subsets of $\mathbf{C} \setminus \Lambda$, one uses the following basic lemma.

2.4. Lemma. *The Eisenstein series $G_k(\Lambda) = \sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-k}$ is absolutely convergent for every integer $k > 2$.*

The proof of 2.4 is elementary. One can estimate the number of lattice points in annuli around the origin, which grows linearly in the 'size' of the annuli. Note that the values $G_k(\Lambda)$ equal zero if $k > 2$ is odd, since then the terms for λ and $-\lambda$ cancel. □

Exercise 3. Prove lemma 2.4.

From the lemma, one deduces that $\wp_\Lambda$ is a well-defined meromorphic function on $\mathbf{C}$ with double poles at the elements of Λ. Some elementary calculus leads to the Laurent expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2}(\Lambda) z^{2n} \tag{2.5}$$

for $\wp(z)$ around the origin. In order to show that $\wp_\Lambda$ is periodic modulo Λ, one notes first that the derivative $\wp'(z) = \sum_{\lambda\in\Lambda}(z-\lambda)^{-3}$ is clearly periodic modulo Λ. For $\wp$ itself, it follows that for $\lambda \in \Lambda$ we have $\wp(z+\lambda) = \wp(z) + c_\lambda$. To prove that we have $c_\lambda = 0$ for generators $\lambda = \lambda_i$ of Λ, and therefore for all λ, we take $z = -\lambda_i/2$ and use that $\wp$ is an *even* function.

A second method to construct periodic functions proceeds multiplicatively, by writing down a convergent Weierstrass product

$$\sigma(z) = \sigma_\Lambda(z) = z \prod_{\lambda\in\Lambda\setminus\{0\}} (1 - \frac{z}{\lambda}) e^{(z/\lambda)+\frac{1}{2}(z/\lambda)^2}$$

for a function having simple zeroes at the points in Λ.

Exercise 4. Show that the product expansion for the σ-function converges uniformly on compact subsets of $\mathbf{C}$. [Hint: pass to the logarithm and use 2.4.]

By 2.3, termwise differentiation of the logarithmic derivative

$$\frac{d \log \sigma(z)}{dz} = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right) \tag{2.6}$$

yields the relation $\frac{d^2 \log \sigma(z)}{dz^2} = -\wp(z)$. As $\wp(z)$ is periodic, we can find $a_\lambda, b_\lambda \in \mathbf{C}$ for each $\lambda \in \Lambda$ such that we have $\sigma(z+\lambda) = e^{a_\lambda z + b_\lambda}\sigma(z)$ for all $z \in \mathbf{C}$. One sometimes says that $\sigma(z)$ is a *theta function* with respect to the lattice Λ.

We are now in a position to finish the proof of 2.1. We still need to show that every divisor $D = \sum_w n_w[w]$ that is of degree 0 and in the kernel of the summation map is the divisor of an elliptic function. Write $\Sigma(D) = \sum_w n_w w = \lambda \in \Lambda$. If λ is non-zero, we add the trivial divisor $[0] - [\lambda]$ to D to obtain a divisor satisfying $\sum_w n_w w = 0$. Now the meromorphic function $f_D = \prod_w \sigma(z-w)^{n_w}$ is elliptic with respect to Λ, as it satisfies

$$f_D(z+\lambda) = e^{a_\lambda \sum_w n_w w + b_\lambda \sum_w n_w} \sigma(z) = f_D(z)$$

for $\lambda \in \Lambda$. By construction, it has divisor $(f_D) = \sum_w n_w[w] = D$. This finishes the proof of 2.1.

The factor group $\mathrm{Jac}(T) = \mathrm{Div}^0(T)/\mathrm{div}[\mathcal{M}(T)^*]$ of divisor classes of degree zero is the *Jacobian* of T. The content of theorem 2.1 may be summarized by the statement that T is canonically isomorphic to its Jacobian.

The actual construction of elliptic functions in the proof of 2.1 shows that the field $\mathcal{M}(T)$ can be given explicitly in terms of functions related to the $\wp$-function. The precise statement is as follows.

2.7. Theorem. *The elliptic function field corresponding to* $T = \mathbf{C}/\Lambda$ *equals*

$$\mathcal{M}(T) = \mathbf{C}(\wp_\Lambda, \wp'_\Lambda).$$

This is a quadratic extension of the field $\mathbf{C}(\wp_\Lambda)$ *of even elliptic functions.*

Proof. Any elliptic function f is the sum $f(z) = \frac{f(z)+f(-z)}{2} + \frac{f(z)-f(-z)}{2}$ of an even and an odd elliptic function, and for odd f the function $f\wp'$ is even. It follows that $\wp'$ generates $\mathcal{M}(T)$ over the field of even elliptic functions, and that this extension is quadratic.

Let $f \in \mathcal{M}(T)^*$ be even. We need to show that f is a rational expression in $\wp = \wp_\Lambda$. We note first that $\mathrm{ord}_w(f)$ is even at '2-torsion points' w satisfying $w = -w \in T$: this follows from the fact that the derivatives of odd order of f are odd elliptic functions, and such functions have non-zero order at a point $w = -w \in T$. We can therefore write

$$(f) = \sum\nolimits_{w\in T} c_w([w] + [-w]) = \sum\nolimits_{w\in T} c_w([w] + [-w] - 2[0]).$$

We can assume that no term with $w = 0$ occurs in the last sum. As the functions f and $\prod_w (\wp(z) - \wp(w))^{c_w}$ have the same divisor, their quotient is a constant. □

Exercise 5. Let $f \in \mathcal{M}(T)$ have polar divisor $2 \cdot (0)$. Prove: $f = c_1\wp + c_2$ for certain $c_1, c_2 \in \mathbf{C}$.

The function $\wp'$ is an odd elliptic function with polar divisor $3 \cdot (0)$, so it is of order 3. Its 3 zeroes are the 3 points $\lambda_1/2$, $\lambda_2/2$ and $\lambda_3/2 = (\lambda_1 + \lambda_2)/2$ of order 2 in $T = \mathbf{C}/\Lambda$. The even function $(\wp')^2$ has divisor $\sum_{i=1}^{3}[2 \cdot (\lambda_i/2) - 2 \cdot (0)]$, so the preceding proof and a look at the first term $4z^{-6}$ of the Laurent expansion of $(\wp')^2$ around 0 show that we have a differential equation

$$(\wp'(z))^2 = 4\prod_{i=1}^{3}(\wp(z) - \wp(\lambda_i/2)). \tag{2.8}$$

The coefficients of the cubic polynomial in 2.8 depend on the lattice Λ in the following explicit way.

2.9. Theorem. *The* $\wp$*-function for* Λ *satisfies a Weierstrass differential equation*

$$(\wp'_\Lambda)^2 = 4\wp_\Lambda^3 - g_2\wp_\Lambda - g_3$$

with coefficients $g_2 = g_2(\Lambda) = 60G_4(\Lambda)$ *and* $g_3 = g_3(\Lambda) = 140G_6(\Lambda)$. *The discriminant* $\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2$ *does not vanish.*

Proof. The derivation of the differential equation is a matter of careful administration based on the Laurent expansion around $z = 0$ in (2.5). From the local expansions $\wp(z) = z^{-2} + 3G_4z^2 + O(z^4)$ and $\wp'(z) = -2z^{-3} + 6G_4z + 20G_6z^3 + O(z^5)$ one easily finds

$$(\wp'(z))^2 = 4z^{-6} - 24G_4z^{-2} - 80G_6 + O(z^2)$$
$$4\wp(z)^3 = 4z^{-6} + 36G_4z^{-2} + 60G_6 + O(z^2).$$

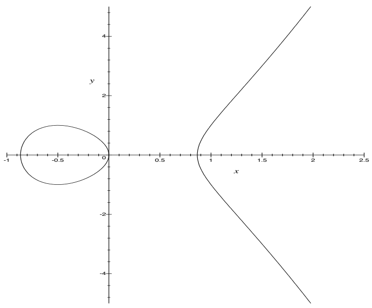
It follows that $(\wp'(z))^2 - 4\wp^3 + 60G_4\wp + 140G_6$ is a holomorphic elliptic function that vanishes at the origin, so it is identically zero. For the non-vanishing of the discriminant

$$\begin{aligned}\Delta(\Lambda) &= g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \\ &= 16 \cdot (\wp(\tfrac{\lambda_1}{2}) - \wp(\tfrac{\lambda_2}{2}))^2 \cdot (\wp(\tfrac{\lambda_1}{2}) - \wp(\tfrac{\lambda_3}{2}))^2 \cdot (\wp(\tfrac{\lambda_2}{2}) - \wp(\tfrac{\lambda_3}{2}))^2,\end{aligned}$$

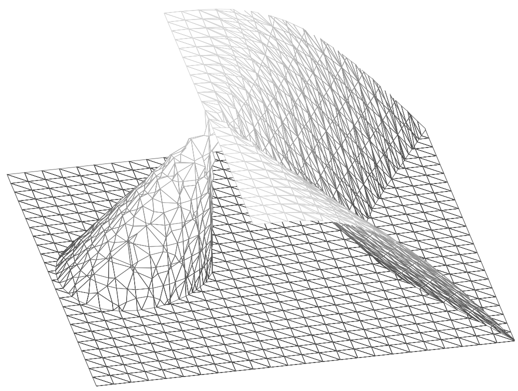
one observes that the function $\wp(z) - \wp(\lambda_i/2)$ is elliptic of order 2 with a double zero at $\lambda_i/2$, so it cannot vanish at $\lambda_j/2$ for $j \neq i$. □

Exercise 6. Show that the non-constant solutions to the differential equation $(y')^2 = 4y^3 - g_2y - g_3$ corresponding to a lattice Λ are the functions $\wp_\Lambda(z - z_0)$ with $z_0 \in \mathbf{C}$. What are the constant solutions?

It follows from 2.9 that the map $W : z \mapsto (\wp(z), \wp'(z))$ maps T to a complex curve in $\mathbf{C}^2$ with equation $y^2 = 4x^3 - g_2x - g_3$. This is exactly the kind of map we have been considering in section 1. If g_2 and g_3 are real, one can sketch the curve in $\mathbf{R}^2$. For a Weierstrass polynomial having three real roots the picture looks as follows.



In order to deal with the poles of the map W, we pass to the *projective completion* of our curve in $\mathbf{P}^2(\mathbf{C})$. This is by definition the zero set in $\mathbf{P}^2(\mathbf{C})$ of the homogenized equation $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$; it consists of the 'affine points' $(x : y : 1)$ coming from the original curve and the 'point at infinity' $(0 : 1 : 0)$. One can view the lines through the origin in $\mathbf{R}^3$ as the points of the real projective plane $\mathbf{P}^2(\mathbf{R})$, and draw the following picture of the completed curve. The point at infinity in this picture is the single line in the plane $Z = 0$.



2.10. Theorem. *Let $\Lambda \subset \mathbf{C}$ be a lattice. Then the Weierstrass map*

$$W: \ z \longmapsto \begin{cases} (\wp(z) : \wp'(z) : 1) & \text{for } z \neq 0 \\ (0 : 1 : 0) & \text{for } z = 0 \end{cases}$$

induces a bijection between the torus $T = \mathbf{C}/\Lambda$ and the complex elliptic curve E_Λ with projective Weierstrass equation

$$E_\Lambda : \quad Y^2 Z = 4X^3 - g_2(\Lambda) X Z^2 - g_3(\Lambda) Z^3.$$

Proof. By 2.9, the torus $\mathbf{C}/\Lambda$ is mapped to the curve E_Λ. We have to show that every affine point $P = (x, y)$ on E_Λ is the image of a unique point $z \in T \setminus \{0\}$. The divisor of the function $\wp(z) - x$ is of the form $(w) + (-w) - 2(0)$ for some $w \in T$ that is determined up to sign. For $w = -w \in T$ we have $y = 0$, and $z = w$ is the unique point mapping to P. Otherwise, we have $\wp'(w) = \pm y \neq 0$, and exactly one of w and $-w$ maps to P. □

2.11. Corollary. *The Weierstrass parametrization* 2.10 *induces a group structure on the set $E_\Lambda(\mathbf{C})$ of points of the elliptic curve E_Λ. The zero element of $E_\Lambda(\mathbf{C})$ is the 'point at infinity' $O_E = (0 : 1 : 0)$, and the inverse of the point $(X : Y : Z)$ is $(X : -Y : Z)$. Any three distinct points in $E_\Lambda(\mathbf{C})$ that are collinear in $\mathbf{P}^2(\mathbf{C})$ have sum O_E.*

Proof. It is clear that $W(0) = O_E$ is the zero element for the induced group structure on $E_\Lambda(\mathbf{C})$, and that the inverse of the point $(\wp(z) : \wp'(z) : 1)$ is $(\wp(-z) : \wp'(-z) : 1) = (\wp(z) : -\wp'(z) : 1)$. It remains to show that three collinear points in $E_\Lambda(\mathbf{C})$ have sum zero. Let $L : aX + bY + cZ = 0$ be the line passing through three such points, and consider the associated elliptic function $f = a\wp + b\wp' + c$. If b is non-zero, the divisor of f is of the form $(f) = (w_1) + (w_2) + (w_3) - 3(0)$ for certain $w_i \in T$. We have $w_1 + w_2 + w_3 = 0 \in T$ by 2.1 (iii), and since the Weierstrass parametrization W maps the w_i to the three points of intersection of L and E_Λ, these points have sum O_E. For $b = 0$ and $a \neq 0$, we are in the case of a 'vertical line' with affine equation $x = -c/a$. The point O_E is on this line. The function $f = a\wp + c$ now has divisor $(f) = (w_1) + (w_2) - 2(0)$, and the same argument as above shows that the 2 affine points of intersection of L and E_Λ are inverse to each other. The case $a = b = 0$ does not occur since then the line L is the line at infinity $Z = 0$, which intersects E_Λ only in O_E. □

Exercise 7. Define multiplicities for the points of intersection of E_Λ with an arbitrary line L, and show that with these multiplicities the 'sum of the points in $L \cap E_\Lambda$' is always equal to O_E.

Corollary 2.11 shows that the group law on $E_\Lambda(\mathbf{C})$ has a simple geometric interpretation. In order to find the sum of 2 points P and Q in $E_\Lambda(\mathbf{C})$, one finds the third point $R = (a, b)$ of intersection of the line through P and Q with E. One then has $P + Q = -R$, so the sum of P and Q equals $(a, -b)$.

From the geometric description, one can derive an explicit addition formula for the points on E_Λ or, equivalently, addition formulas for the functions $\wp$ and $\wp'$. Let $P = (\wp(z_1), \wp'(z_1))$ and $Q = (\wp(z_2), \wp'(z_2))$ be points on E_Λ. If P and Q are inverse to each other, we have $z_1 = -z_2 \bmod \Lambda$ and $P + Q$ is the infinite point O_E. Otherwise, the affine line through P and Q is of the form $y = \lambda x + \mu$ with

$$\lambda = \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} = \frac{4\wp(z_1)^2 + 4\wp(z_1)\wp(z_2) + 4\wp(z_2)^2 - g_2}{\wp'(z_1) + \wp'(z_2)}.$$

The second expression, which is obtained by multiplication of numerator and denominator of the first expression by $\wp'(z_1) + \wp'(z_2)$ and applying 2.9, is also well-defined for $P = Q$; in this case it yields the slope of the tangent line in P. As the cubic equation

$$4x^3 - g_2x - g_3 - (\lambda x + \mu)^2 = 0$$

has roots $\wp(z_1)$, $\wp(z_2)$ and $\wp(z_1 + z_2)$, we find the x coordinate of $P + Q$ to be

$$\text{(2.12)} \qquad \wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 \qquad (z_1 \neq \pm z_2 \bmod \Lambda).$$

In the case $P = Q$, one can use the second expression for λ to find the x-coordinate $\wp(2z_1)$ of $2P$ as a rational function in $\wp(z_1)$.

Exercise 8. Write $\wp(2z)$ as a rational function in $\wp(z)$. Show that this duplication formula for the $\wp$-function also follows from the limit form

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$$

of 2.12 and the differential equation $\wp'' = 6\wp^2 - \frac{1}{2}g_2$, which is obtained by differentiating 2.9.

As in the previous section, we find that the addition formulas on the elliptic curve E_Λ are *algebraic formulas* involving the coefficients g_2 and g_3 of the defining Weierstrass equation. We say that an elliptic curve E with Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ is *defined over* a subfield $K \subset \mathbf{C}$ if g_2 and g_3 are in K. If E is defined over a field $K \subset \mathbf{C}$, the set $E(K)$ of K-valued points is a subgroup of $E(\mathbf{C})$. We will especially be interested in the case where K is the field of rational numbers. When working over $\mathbf{Q}$, it is often convenient to choose variables $X = 4x$ and $Y = 4y$ satisfying the equation $Y^2 = X^3 - 4g_2X - 16g_3$.

In this case the determination of the group $E(\mathbf{Q})$ is a highly non-trivial problem that has its roots in antiquity. The observation that two (not necessarily distinct) points on a cubic curve can be used to find a third point already goes back to Diophantus. His method, which is basically a method for adding points, is known as the *chord-tangent method*.

Exercises.

9. Let f be a non-constant meromorphic function on $\mathbf{C}$. A number $\lambda \in \mathbf{C}$ is said to be a *period* of f if $f(z+\lambda) = f(z)$ for all $z \in \mathbf{C}$. Let Λ be the set of periods of f.
   a. Prove that Λ is a discrete subgroup of $\mathbf{C}$.
   b. Deduce that Λ is of one of the three following forms:
$$\Lambda = \{0\} \qquad \Lambda = \mathbf{Z}\lambda \ \ (\lambda \neq 0) \qquad \Lambda = \mathbf{Z}\lambda_1 \oplus \mathbf{Z}\lambda_2 \ \ (\text{with } \mathbf{C} = \mathbf{R}\lambda_1 + \mathbf{R}\lambda_2)$$

10. Let $\wp$ be the $\wp$-function associated to Λ. Show that the function $z \mapsto e^{\wp(z)}$ is holomorphic on $\mathbf{C} \setminus \Lambda$ and periodic modulo Λ, but not elliptic.

11. Let f be a meromorphic function with non-zero period λ and define $q = q(z) = e^{2\pi i z/\lambda}$. Prove that there exists a meromorphic function $\widehat{f}$ on $\mathbf{C}^*$ satisfying $f(z) = \widehat{f}(q)$, and show that we have $\operatorname{ord}_q(\widehat{f}) = \operatorname{ord}_z(f)$ for all $z \in \mathbf{C}$.

12. Let Λ be a lattice and $\wp$ and σ the associated complex functions. Prove the identity
$$\wp(z) - \wp(a) = -\frac{\sigma(z-a)\sigma(z+a)}{\sigma(a)^2\sigma(z)^2} \qquad (a \notin \Lambda).$$

13. *(Degeneracy of the $\wp$-function.)* Let λ be an element in $\mathbf{C} \setminus \mathbf{R}$ and t a real number.
   a. Prove the identities
$$\lim_{t\to\infty} \wp_{[t,\lambda t]}(z) = \frac{1}{z^2} \qquad \text{and} \qquad \lim_{t\to\infty} \wp_{[1,\lambda t]}(z) = \frac{1}{\sin^2(\pi z)} + \frac{3}{\pi^2}$$
   for $z \in \mathbf{C}^*$ and $z \in \mathbf{C} \setminus \mathbf{Z}$, respectively.
   b. What are the degenerate forms of the function $\sigma(z)$ corresponding to the two cases above, and which identities replace the one in the previous exercise?
   c. Find the degenerate analogues of 2.10, and explain why these two forms of degeneracy are called *additive* and *multiplicative*, respectively.

14. Determine the general solution of the Weierstrass differential equation $(y')^2 = 4y^3 - g_2 y - g_3$ in the degenerate case $g_2^3 = 27g_3^2$.

15. Show that the derivative of the $\wp$-function satisfies $\wp'(z) = -\frac{\sigma(2z)}{\sigma(z)^4}$.

16. Let $\Lambda = [\lambda_1, \lambda_2]$ be a lattice with associated Weierstrass function $\wp$, and consider the Weierstrass functions $\wp_1$ and $\wp_2$ associated to the lattices $\Lambda_1 = \frac{1}{2}\Lambda$ and $\Lambda_2 = [\frac{1}{2}\lambda_1, \lambda_2]$. Prove the identities
$$\wp_1(z) = 4\wp(2z) \qquad \text{and} \qquad \wp_2(z) = \wp(z) + \wp(z + \tfrac{1}{2}\lambda_1) - \wp(\tfrac{1}{2}\lambda_1).$$
   What are the corresponding identities for $\wp_1'$ and $\wp_2'$?

17. Prove: $4\wp(2z) = \wp(z) + \wp(z + \frac{1}{2}\lambda_1) + \wp(z + \frac{1}{2}\lambda_2) + \wp(z + \frac{1}{2}\lambda_3)$.

18. Define the *Weierstrass ζ-function* for the lattice $\Lambda = \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2$ in $\mathbf{C}$ as in (2.6) by $\zeta(z) = \frac{d}{dz}\log\sigma(z)$.

a. Show that there exists a linear function $\eta : \Lambda \to \mathbf{C}$ such that $\zeta(z+\lambda) = \zeta(z) + \eta(\lambda)$ for $\lambda \in \Lambda$ and $z \in \mathbf{C}$, and that we have $\eta(\lambda) = 2\zeta(\lambda/2)$ for $\lambda \notin 2\Lambda$.

The numbers $\eta_i = \eta(\lambda_i)$ $(i = 1, 2)$ are the *quasi-periods* of $\zeta(z)$.

b. Prove the *Legendre relation* $\eta_1\lambda_2 - \eta_2\lambda_1 = \pm 2\pi i$.
[Hint: the right hand side equals $\oint \zeta(z)dz$ around a fundamental parallelogram.]

c. Prove: $\sigma(z+\lambda) = \pm e^{\eta(\lambda)(z+\lambda/2)}\sigma(z)$.

19. *(Weil reciprocity law.)* For an elliptic function f and a divisor $D = \sum_{w \in T} n_w \cdot (w) \in \mathrm{Div}(T)$ on the complex torus T, we let $f(D) = \prod_w f(w)^{n_w} \in \mathbf{C}$. Prove that for any two elliptic functions f and g with disjoint divisors, we have

$$f((g)) = g((f)).$$

[Hint: write f and g as products of σ-functions.]

20. Let $G_k = \sum_{\lambda \in \Lambda'} \lambda^{-k}$ be the Eisenstein series of order k, and define $G_2 = G_1 = 0$ and $G_0 = -1$.

a. Show that $(k-1)(k-2)(k-3)G_k = 6\sum_{j=0}^{k}(j-1)(k-j-1)G_jG_{k-j}$ for all $k \geq 6$.
[Hint: $\wp'' = 6\wp^2 - 30G_4$.]

b. Show that $G_8 = \frac{3}{7}G_4^2$, $G_{10} = \frac{5}{11}G_4G_6$ and $G_{12} = \frac{25}{143}G_6^2 + \frac{18}{143}G_4^3$ and that, more generally, every Eisenstein series can be computed recursively from G_4 and G_6 by the formula

$$(k^2-1)(k-6)G_k = 6\sum_{j=4}^{k-4}(j-1)(k-j-1)G_jG_{k-j}.$$

21. Let Λ be a lattice for which $g_2(\Lambda)$ and $g_3(\Lambda)$ are real. Prove that Λ is either a rectangular lattice spanned by a real and a totally imaginary number, or a rhombic lattice spanned by a real number λ_1 and number λ_2 satisfying $\lambda_2 + \overline{\lambda_2} = \lambda_1$. Show that these cases can be distinguished by the sign of $\Delta(\Lambda)$, and that we have group isomorphisms

$$E_\Lambda(\mathbf{R}) \cong \begin{cases} \mathbf{R}/\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \text{for } \Delta(\Lambda) > 0; \\ \mathbf{R}/\mathbf{Z} & \text{for } \Delta(\Lambda) < 0. \end{cases}$$

22. Let $L(nO)$ be the vector space of meromorphic functions on the torus $T = \mathbf{C}/\Lambda$ having a pole of order at most n in O. Prove:

$$\dim_{\mathbf{C}}(L(nO)) = \begin{cases} n & \text{for } n > 0; \\ 1 & \text{for } n = 0. \end{cases}$$

23. *(Riemann-Roch for the torus.)* For a divisor D on the torus T, let $L(D)$ be the vector space consisting of $f = 0$ and the meromorphic functions $f \neq 0$ on T for which the divisor $(f) + D$ is without polar part. Prove:

$$\dim_{\mathbf{C}}(L(D)) = \begin{cases} \deg(D) & \text{for } \deg(D) > 0; \\ 0 & \text{for } \deg(D) < 0. \end{cases}$$

What can you say if D is of degree 0?

3. Complex elliptic curves

We have seen in the previous section that every complex torus $T = \mathbf{C}/\Lambda$ is 'isomorphic' to the elliptic curve E_Λ with Weierstrass equation $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$. The *uniformization theorem* 3.8 in this section states that conversely, every complex Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ of non-zero discriminant $\Delta = g_2^3 - 27g_3^2$ comes from a torus. This correspondence is actually an *equivalence of categories*. In order to make this into a meaningful statement, we have to define the maps in the categories of complex tori and complex elliptic curves, respectively.

We will first define a set $\mathrm{Hom}(T_1, T_2)$ of maps between complex tori called *isogenies* and study its structure. At the end of this section, we will describe the corresponding algebraic maps between complex Weierstrass curves, which are again called isogenies. These maps will turn out be an important tool in studying the arithmetic of elliptic curves over $\mathbf{Q}$.

3.1. Lemma. *Let $\psi : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ be a continuous map between complex tori. Then there exists a continuous map $\phi : \mathbf{C} \to \mathbf{C}$ such that the diagram*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\phi} & \mathbf{C} \\ \big\downarrow \text{can} & & \big\downarrow \text{can} \\ \mathbf{C}/\Lambda_1 & \xrightarrow{\psi} & \mathbf{C}/\Lambda_2 \end{array}$$

commutes. The map ϕ is uniquely determined up to an additive constant in Λ_2.

Proof. Choose $\phi(0)$ such that the diagram commutes for $z = 0$. If $z \in \mathbf{C}$ is arbitrary, choose a path $\gamma : 0 \to z$ in $\mathbf{C}$. Let $\overline{\gamma} : \psi(\bar{0}) \to \psi(\bar{z})$ be the path in $\mathbf{C}/\Lambda_2$ obtained by reducing modulo Λ_1 and applying ψ. As the natural map $\mathbf{C} \to \mathbf{C}/\Lambda_2$ is a covering map, $\overline{\gamma}$ can uniquely be lifted under this map to a path in $\mathbf{C}$ starting in $\phi(0)$, and we define $\phi(z)$ as the end point of this map. The value $\phi(z)$ is independent of the choice of the path γ since $\mathbf{C}$ is simply connected, and it is clear that ϕ is continuous. If ϕ' is another map for which the diagram commutes, then their difference $\phi - \phi'$ is a continuous map $\mathbf{C} \to \Lambda_2$, so it is constant. □

If the map ϕ in lemma 3.1 is a holomorphic function, we call ψ an *analytic map* between the tori. An analytic map $\psi : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ is called an *isogeny* if it satisfies $\psi(0) = 0$. An analytic map ψ is the composition of the isogeny $\psi - \psi(0)$ with a translation over $\psi(0)$.

3.2. Theorem. *Let $\psi : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ be an isogeny. Then there exists $\alpha \in \mathbf{C}$ such that we have*

$$\psi(z \bmod \Lambda_1) = \alpha z \bmod \Lambda_2 \qquad \textit{and} \qquad \alpha\Lambda_1 \subset \Lambda_2.$$

Conversely, every $\alpha \in \mathbf{C}$ satisfying $\alpha\Lambda_1 \subset \Lambda_2$ gives rise to an isogeny $\mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$.

Proof. Let $\phi : \mathbf{C} \to \mathbf{C}$ be the lift of ψ satisfying $\phi(0) = 0$. For every $\lambda_1 \in \Lambda_1$, the holomorphic function $\phi(z) - \phi(z + \lambda_1)$ has values in Λ_2, so it is constant. It follows that $\phi'(z)$ is a holomorphic function with period lattice Λ_1, so by Liouville's theorem it is constant. For ϕ itself we find $\phi(z) = \alpha z$ for some $\alpha \in \mathbf{C}$. As Λ_1 maps to zero in $\mathbf{C}/\Lambda_2$, we have $\alpha\Lambda_1 \subset \Lambda_2$. Conversely, it is clear that any α of this form induces an isogeny. □

3.3. Corollary. *Every complex isogeny is a homomorphism on the group of points. The set of isogenies* $\mathrm{Hom}(\mathbf{C}/\Lambda_1, \mathbf{C}/\Lambda_2)$ *carries a natural group structure.* □

We say that two complex tori are *isogenous* if there exists a non-zero isogeny between them. Note that a non-zero isogeny is always surjective.

Exercise 1. Take $\Lambda_1 = \mathbf{Z} + \mathbf{Z}\, i$ and $\Lambda_2 = \mathbf{Z} + \mathbf{Z}\, i\pi$. Prove: $\mathrm{Hom}(\mathbf{C}/\Lambda_1, \mathbf{C}/\Lambda_2) = 0$.

For a non-zero isogeny $\psi : T_1 = \mathbf{C}/\Lambda_1 \to T_2 = \mathbf{C}/\Lambda_2$, we define the *degree* of ψ as

$$\deg(\psi) = \# \ker \psi = \#[(\alpha^{-1}\Lambda_2) \bmod \Lambda_1] = [\Lambda_2 : \alpha\Lambda_1].$$

The degree of the zero isogeny is by definition equal to 0.

For ψ of degree $n > 0$, we have inclusions of lattices $n\Lambda_2 \overset{n}{\subset} \alpha\Lambda_1 \overset{n}{\subset} \Lambda_2$. This shows that multiplication by n/α maps Λ_2 to a lattice of index n in Λ_1. The corresponding isogeny $\widehat{\psi} : T_2 \to T_1$ is the *dual isogeny* corresponding to ψ. Note that $\widehat{\psi} \circ \psi$ and $\psi \circ \widehat{\psi}$ are multiplication by n on T_1 and T_2, respectively.

Exercise 2. Show that being isogenous is an equivalence relation on the set of complex tori, and that there are uncountably many isogeny classes of complex tori.

Two complex tori $\mathbf{C}/\Lambda_1$ and $\mathbf{C}/\Lambda_2$ are isomorphic if there is an invertible isogeny between them, i.e., an isogeny of degree 1. This happens if and only if $\Lambda_2 = \alpha\Lambda_1$ for some $\alpha \in \mathbf{C}^*$. In that case we say that Λ_1 and Λ_2 are isomorphic or *homothetic*. For homothetic lattices Λ_1 and Λ_2 we have $g_2(\Lambda_2) = \alpha^{-4} g_2(\Lambda_1)$ and $g_3(\Lambda_2) = \alpha^{-6} g_3(\Lambda_1)$ for some α, so the *j-invariant*

$$j(\Lambda) = 1728 \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27 g_3(\Lambda)^2} = 1728 \frac{g_2(\Lambda)^3}{\Delta(\Lambda)}$$

of a lattice is defined on isomorphism classes of lattices. Note that $j(\Lambda)$ is well-defined since $\Delta(\Lambda)$ does not vanish. The factor $1728 = 12^3$ is traditional; it is related to the Fourier expansion of the j-function.

3.4. Lemma. *Two lattices are homothetic if and only if their j-invariants coincide.*

Proof. We still need to show that the equality $j(\Lambda_1) = j(\Lambda_2)$ implies that Λ_1 and Λ_2 are homothetic. From the equality $j(\Lambda_1) = j(\Lambda_2)$ we easily derive that there exists $\alpha \in \mathbf{C}^*$ such that we have $g_2(\Lambda_2) = \alpha^{-4} g_2(\Lambda_1)$ and $g_3(\Lambda_2) = \alpha^{-6} g_3(\Lambda_1)$. Then Λ_2 and $\alpha\Lambda_1$ have the same values of g_2 and g_3, so the $\wp$-functions $\wp_{\Lambda_2}$ and $\wp_{\alpha\Lambda_1}$ coincide. In particular, their sets of poles Λ_2 and $\alpha\Lambda_1$ coincide. □

Every lattice $\Lambda = [\lambda_1, \lambda_2]$ is homothetic to a lattice $[1, z]$ with $z = \lambda_2/\lambda_1$ in the complex upper half plane, so we can view j as a function $j : \mathcal{H} \to \mathbf{C}$. The Eisenstein series $G_k(z) = G_k([1, z])$ are holomorphic on $\mathcal{H}$ by 2.4, so j is again a holomorphic function on $\mathcal{H}$.

Two lattices $[1, z_1]$ and $[1, z_2]$ are homothetic if and only if we have $z_2 = \frac{az_1+b}{cz_1+d}$ for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$. The identity

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = (ad-bc)\frac{\mathrm{Im}(z)}{|cz+d|^2} \tag{3.5}$$

shows that only the matrices in $\mathrm{SL}_2(\mathbf{Z})$ map $\mathcal{H}$ to itself. We conclude that $j : \mathcal{H} \to \mathbf{C}$ is constant on $\mathrm{SL}_2(\mathbf{Z})$-orbits, and that the induced function $j : \mathrm{SL}_2(\mathbf{Z}) \setminus \mathcal{H} \to \mathbf{C}$ on the orbit space is injective.

3.6. Theorem. *The map* $j : \mathrm{SL}_2(\mathbf{Z}) \setminus \mathcal{H} \to \mathbf{C}$ *is a bijection.*

The main ingredient in the proof of 3.6 is the construction of a fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{H}$. The following statement is sufficient for our purposes.

3.7. Lemma. *Every* $\mathrm{SL}_2(\mathbf{Z})$*-orbit in* $\mathcal{H}$ *has a representative in the set*

$$D = \{z \in \mathcal{H} : |z| \geq 1 \quad \text{and} \quad -1/2 \leq \mathrm{Re}(z) < 1/2\}.$$

Proof. Pick $z \in \mathcal{H}$. As the elements $cz+d$ with $c, d \in \mathbf{Z}$ form a lattice in $\mathbf{C}$, the numerator $|cz+d|^2$ in (3.5) is bounded from below, so there exists an element z_0 in the orbit of z for which $\mathrm{Im}(z)$ is maximal. Applying a translation matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ mapping z_0 to $z_0 + k$ when necessary, we may assume that $\mathrm{Re}(z_0)$ is in $[-1/2, 1/2)$. From the inequality $\mathrm{Im}(-1/z_0) = |z_0|^{-2}\mathrm{Im}(z_0) \leq \mathrm{Im}(z_0)$ we find $|z_0| \geq 1$, so z_0 is in D. ☐

Exercise 3. Find a representative in D for the $\mathrm{SL}_2(\mathbf{Z})$-orbit of $\frac{1+2i}{100}$.

Proof of 3.6. It remains to show that the image $j[\mathcal{H}]$ of the j-function is all of $\mathbf{C}$. As j is a non-constant holomorphic function on $\mathcal{H}$, its image $j[\mathcal{H}]$ is open in $\mathbf{C}$. We will show that $j[\mathcal{H}]$ is also closed in $\mathbf{C}$. By the connectedness of $\mathbf{C}$, this proves what we want.

Let $j = \lim_{n\to\infty} j(z_n)$ be a limit point of $j[\mathcal{H}]$ in $\mathbf{C}$. By picking the z_n suitably inside their $\mathrm{SL}_2(\mathbf{Z})$-orbit, we may assume that all z_n lie in D. If the values of $\mathrm{Im}(z_n)$ remain bounded, the sequence $\{z_n\}_n$ lies in a bounded subset of D, and we can pick any limit point $z \in \mathcal{H}$ of the sequence to find $j(z) = j \in j[\mathcal{H}]$.

If the values of $\mathrm{Im}(z_n)$ are not bounded, we can pass to a subsequence and assume $\lim_{n\to\infty} \mathrm{Im}(z_n) = +\infty$. From the definition of g_2 and g_3 in theorem 2.9 we now find

$$\lim_{n\to\infty} g_2(z_n) = 60 \cdot 2 \sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{4\pi^4}{3} \qquad \text{and} \qquad \lim_{n\to\infty} g_3(z_n) = 140 \cdot 2 \sum_{m=1}^{\infty} \frac{1}{m^6} = \frac{8\pi^6}{27},$$

so $\Delta(z_n) = g_2(z_n)^3 - 27g_3(z_n)^2$ tends to 0. This implies $\lim_{n\to\infty} |j(z_n)| = +\infty$, contradicting the assumption that $j(z_n)$ converges. ☐

The main corollary of 3.6 is the following theorem. It enables us to translate many statements over complex elliptic curves into analytic facts.

3.8. Uniformization theorem. *Given any two integers $g_2, g_3 \in \mathbf{C}$ with $g_2^3 - 27g_2^2 \neq 0$, there exists a lattice $\Lambda \subset \mathbf{C}$ with $g_2(\Lambda) = g_2$ and $g_3(\Lambda) = g_3$. In particular, every complex elliptic curve comes from a complex torus in the sense of* 2.7.

Proof. Pick a lattice Λ with j-invariant $j(\Lambda) = g_2^3/(g_2^3 - 27g_3^2)$. As in the proof of 3.4, we find that there exists $\alpha \in \mathbf{C}$ satisfying $g_2(\Lambda) = \alpha^4 g_2$ and $g_3(\Lambda) = \alpha^6 g_3$. Now the lattice $\alpha\Lambda$ does what we want. □

A complex elliptic curve E in Weierstrass form, or briefly *Weierstrass curve*, can be specified as a pair (g_2, g_3) of coefficients in the corresponding equation $y^2 = 4x^3 - g_2x - g_3$. We require that the discriminant $\Delta(E) = g_2^3 - 27g_3^2$ does not vanish and define the j-invariant of E as $j(E) = 1728\, g_2^3/\Delta(E)$. Weierstrass curves are said to be *isomorphic* if their j-invariants coincide. As we have already seen, Weierstrass curves with coefficients (g_2, g_3) and (g_2', g_3') are isomorphic if and only if there exists $\alpha \in \mathbf{C}$ satisfying $g_2' = \alpha^4 g_2$ and $g_3' = \alpha^6 g_3$.

Exercise 4. Show that a Weierstrass curve E is isomorphic to a Weierstrass curve defined over $\mathbf{Q}(j(E))$.

An *isogeny* between Weierstrass curves is for us simply a map coming from an isogeny between the corresponding complex tori. Its degree is the degree of the corresponding isogeny between tori. With this definition, the categories of complex tori and the category of Weierstrass curves, each with the isogenies as their morphisms, become equivalent in view of 3.8.

Our definition of an isogeny $\psi : E \to \widetilde{E}$ between curves parametrized by $\mathbf{C}/\Lambda$ and $\mathbf{C}/\widetilde{\Lambda}$ means that ψ fits in a commutative diagram

$$\begin{array}{ccc} \mathbf{C}/\Lambda & \xrightarrow{z \to \alpha z} & \mathbf{C}/\widetilde{\Lambda} \\ \Big\downarrow W & & \Big\downarrow \widetilde{W} \\ E & \xrightarrow{\psi} & \widetilde{E}. \end{array}$$

Here W and $\widetilde{W}$ denote the Weierstrass parametrizations, and $\alpha \in \mathbf{C}$ satisfies $\alpha\Lambda \subset \widetilde{\Lambda}$. We see that ψ can be described in terms of Weierstrass $\wp$-functions as

$$\psi : (\wp(z), \wp'(z)) \longmapsto (\widetilde{\wp}(\alpha z), \widetilde{\wp}'(\alpha z)).$$

As $z \mapsto \widetilde{\wp}(\alpha z)$ and $z \mapsto \widetilde{\wp}'(\alpha z)$ are elliptic functions on $\mathbf{C}/\Lambda$, they are rational expressions in $\wp(z)$ and $\wp'(z)$. Thus ψ is actually an *algebraic map* $E \to \widetilde{E}$ that is everywhere defined. It is a *morphism* of curves in the sense of algebraic geometry.

3.9. Theorem. *Let $\psi : E \to \widetilde{E}$ be an isogeny of degree $n > 0$ between Weierstrass curves. Then there exist $\alpha \in \mathbf{C}$ and monic coprime polynomials $A, B \in \mathbf{C}[X]$ of degree n and $n-1$, respectively, such that ψ is given on the affine points of E by the algebraic map*

$$\psi : (x, y) \longmapsto \left(\frac{A(x)}{\alpha^2 B(x)}, \frac{A'(x)B(x) - A(x)B'(x)}{\alpha^3 B(x)^2}\, y \right).$$

Proof. We may suppose that ψ corresponds to a diagram as above. As $\wp(\alpha z)$ is even and periodic modulo Λ, there exist $c \in \mathbf{C}$ and monic coprime polynomials $A, B \in \mathbf{C}[X]$, say of degree a and b, for which we have the identity

$$\widetilde{\wp}(\alpha z) = c\,\frac{A(\wp(z))}{B(\wp(z))}.$$

Comparison of the orders and leading coefficients of the poles of these functions in $z = 0$ yields equalities $c = \alpha^{-2}$ and $2 = 2a - 2b$; in particular we have $a = b + 1$. Now consider the commutative diagram

$$\begin{array}{ccc} \mathbf{C}/\Lambda & \stackrel{\psi}{\longrightarrow} & \mathbf{C}/\widetilde{\Lambda} \\ \downarrow{\scriptstyle\wp} & & \downarrow{\scriptstyle\widetilde{\wp}} \\ \mathbf{P}^1(\mathbf{C}) & \stackrel{\psi_x}{\longrightarrow} & \mathbf{P}^1(\mathbf{C}), \end{array}$$

in which we write ψ again for the isogeny between tori corresponding to ψ. By definition of the degree, ψ is n to 1. The vertical maps are *generically* 2 to 1, meaning that for all but finitely many $x \in \mathbf{P}^1(\mathbf{C})$, the fibers $\wp^{-1}(x)$ and $\widetilde{\wp}^{-1}(x)$ consist of 2 elements. This implies that the composition $\widetilde{\wp} \circ \psi$ is generically $2n$ to 1, and consequently the map $\psi_x : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$, which maps x to $A(x)/(\alpha^2 B(x))$, is generically n to 1. This easily yields $a = n$, as desired. Differentiation of the identity for $\widetilde{\wp}(\alpha z)$ with respect to z yields the value of the y-coordinate of ψ. □

Exercise 5. Let $A, B \in \mathbf{C}[X]$ be coprime polynomials of degree a and b. Show that the map on $\mathbf{P}^1(\mathbf{C})$ defined by $x \mapsto A(x)/B(x)$ is generically $\max(a, b)$ to 1.

3.10. Example. Let $\Lambda = [\lambda_1, \lambda_2]$ be any lattice, and define $\widetilde{\Lambda} = [\frac{1}{2}\lambda_1, \lambda_2]$. Then Λ is of index 2 in $\widetilde{\Lambda}$, and the natural map $T = \mathbf{C}/\Lambda \to \widetilde{T} = \mathbf{C}/\widetilde{\Lambda}$ is an isogeny of degree 2. Its kernel is generated by the 2-torsion element $\frac{1}{2}\lambda_1 \in \mathbf{C}/\Lambda$. On the associated Weierstrass curve $E : y^2 = 4x^3 - g_2 x - g_3$, this corresponds to a point of the form $(a, 0)$. The equation can be written correspondingly as $y^2 = (x - a)(4x^2 + 4ax + \frac{g_3}{a})$.

In order to find the polynomials A and B from 3.9 in this case, we have to express the Weierstrass function $\widetilde{\wp}(z)$ associated to $\widetilde{\Lambda}$ as a rational function in the Weierstrass function $\wp(z)$ associated to Λ. From exercise 2.16, we have the useful identity

$$\widetilde{\wp}(z) = \wp(z) + \wp(z + \tfrac{1}{2}\lambda_1) - \wp(\tfrac{1}{2}\lambda_1).$$

It is now straightforward from the addition formula (2.12) to evaluate

$$\widetilde{\wp} = -2a + \frac{\wp'^2}{4(\wp - a)^2} = -2a + \frac{\wp^2 + a\wp + \frac{g_3}{4a}}{\wp - a} = \frac{\wp^2 - a\wp + 2a^2 + \frac{g_3}{4a}}{\wp - a}.$$

As expected, A and B are monic of degrees 2 and 1. Rewriting $\frac{g_3}{4a} = a^2 - \frac{g_2}{4}$, we can write the complete isogeny in algebraic terms as

$$(x, y) \longmapsto (\widetilde{x}, \widetilde{y}) = \left(x + \frac{12a^2 - g_2}{4(x - a)}, \big(1 - \frac{12a^2 - g_2}{4(x - a)^2}\big) y \right).$$

We refer to the exercises for a proof that $(\widetilde{x}, \widetilde{y})$ is a point on the Weierstrass curve $\widetilde{E}$ with equation $y^2 = 4(x + 2a)(x^2 - 2ax + g_2 - 11a^2)$.

Exercise 6. Show that the isogeny in 3.10 is given by $(x, y) \mapsto (x + x_T - a, y + y_T)$, where $(x_T, y_T) = (x, y) + (a, 0)$ in the group $E(\mathbf{C})$.

Theorem 3.9 shows that isogenies between elliptic curves, which we defined originally as *analytic maps* between tori, turn out be *algebraic maps*, i.e., given by rational functions in the coordinates. Conversely, one can show that all algebraic maps between Weierstrass curves are analytic, so that algebraic and analytic maps come down to the same thing. This equivalence is a simple example of a 'GAGA-phenomenon', an abbreviation referring to a 1956 paper of Serre, *Géométrie algébrique et géométrie analytique*, which is devoted to similar equivalences.

An even simpler example of the phenomenon indicated above is the classification in theorem 2.7 of the meromorphic functions on a torus T. Such functions, which are by definition analytic maps $T \to \mathbf{P}^1(\mathbf{C})$, turn out to be rational functions in the coordinates when viewed as maps on the associated Weierstrass curve. The function field $\mathcal{M}(T) = \mathbf{C}(\wp, \wp')$ of T is therefore isomorphic to the *function field* $\mathcal{M}(E)$ of rational functions in the affine coordinates on E. This field is usually defined as the field of fractions of the *coordinate ring* $\mathbf{C}[x, y]/(y^2 - 4x^3 + g_2x + g_3)$, which is the ring of polynomial functions on the affine part of E. From an algebraic point of view, $\mathcal{M}(E)$ is a quadratic extension $\mathbf{C}(x, \sqrt{4x^3 - g_2x - g_3})$ of the rational function field $\mathbf{C}(x)$.

The function field $\mathcal{M}(\mathbf{P}^1(\mathbf{C}))$ of meromorphic functions on the Riemann sphere is also algebraic: it is the rational function field $\mathbf{C}(x)$.

Every isogeny $\psi : T \to \widetilde{T}$ between complex tori induces a map $\psi^* : \mathcal{M}(\widetilde{T}) \to \mathcal{M}(T)$ in the opposite direction mapping an elliptic function $f \in \mathcal{M}(\widetilde{T})$ to $f \circ \psi$. If ψ is non-zero, this is an injective homomorphism of fields.

3.11. Theorem. *Let $\psi : T \to \widetilde{T}$ be an isogeny of degree $n > 0$. Then the field extension $\psi^*[\mathcal{M}(\widetilde{T})] \subset \mathcal{M}(T)$ is an algebraic extension of degree n.*

Proof. As $\mathcal{M}(T)$ and $\mathcal{M}(\widetilde{T})$ are quadratic extensions of $\mathbf{C}(\wp)$ and $\mathbf{C}(\widetilde{\wp})$, respectively, it suffices to show that $\mathbf{C}(\wp)$ is algebraic of degree n over $\psi^*[\mathbf{C}(\widetilde{\wp})]$. In view of 3.9, this follows from the following lemma. □

3.12. Lemma. *Let $A, B \in \mathbf{C}[X]$ be coprime polynomials of degree a and b. If A and B are not both constant, then $\mathbf{C}(x)$ is an algebraic of degree $\max(a, b)$ of $\mathbf{C}(\frac{A(x)}{B(x)})$.*

Proof. Write $Y = \frac{A(x)}{B(x)}$, then x is a zero of the polynomial $F = A(X) - YB(X) \in \mathbf{C}[X, Y]$ of degree $\max(a, b)$ in X with coefficients in $\mathbf{C}(Y)$.It remains to show that F is irreducible. As F is of degree 1 in Y, it can only be reducible if there is a polynomial in $\mathbf{C}[X] \setminus \mathbf{C}$ dividing it; this is excluded by the coprimality assumption on A and B. □

It is a general fact from algebraic geometry that degrees of maps can be read off from the degrees of the corresponding function field extension. Over $\mathbf{C}$ or $\overline{\mathbf{Q}}$, the degree of a map is the cardinality of all but finitely many fibers.

Exercise 7. Check this fact for the projections π_x and π_y of a Weierstrass curve E on the axes. *Can you generalize the argument to arbitrary rational functions $E \to \mathbf{P}^1(\mathbf{C})$?

Exercises.

8. The *multiplicator ring* of a lattice Λ is defined as $\mathcal{O} = \mathcal{O}(\Lambda) = \{\alpha \in \mathbf{C} : \alpha\Lambda \subset \Lambda\}$. Show that $\mathcal{O}$ is a subring of $\mathbf{C}$ isomorphic to the endomorphism ring $\mathrm{End}(\mathbf{C}/\Lambda)$ of the torus $\mathbf{C}/\Lambda$. Show also that we have $\mathcal{O}(\Lambda) = \mathbf{Z}$ unless Λ is homothetic to a lattice of the form $[1, \lambda]$, with $\lambda \in \mathbf{C} \setminus \mathbf{R}$ the zero of an irreducible quadratic polynomial $aX^2 + bX + c \in \mathbf{Z}[X]$, and that in this exceptional case we have $\mathcal{O}(\Lambda) = \mathbf{Z}[\frac{D+\sqrt{D}}{2}]$ with $D = b^2 - 4ac < 0$.
[In the exceptional case, we say that $\mathbf{C}/\Lambda$ has *complex multiplication* by $\mathcal{O}$.]

9. Show that the subrings of $\mathbf{C}$ that are lattices correspond bijectively to the set of negative integers $D \equiv 0, 1 \bmod 4$ under the map $D \mapsto \mathcal{O}(D) = \mathbf{Z}[\frac{D+\sqrt{D}}{2}]$. Show that there exists a ring homomorphism $\mathcal{O}(D_1) \to \mathcal{O}(D_2)$ if and only if D_1/D_2 is a square in $\mathbf{Z}$.
[One calls $\mathcal{O}(D)$ the *quadratic order of discriminant* D.]

*10. Show that the isomorphism classes of complex tori with complex multiplication by $\mathcal{O}$ correspond bijectively to the elements of the Picard group $\mathrm{Pic}(\mathcal{O})$ of $\mathcal{O}$.

11. Show that the degree map $\deg : \mathrm{End}(\mathbf{C}/\Lambda) \to \mathbf{Z}$ is a multiplicative function, and that there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{End}(\mathbf{C}/\Lambda) & \xrightarrow{\sim} & \mathcal{O}(\Lambda) & \subset & \mathbf{C} \\ \downarrow{\scriptstyle \deg} & & \downarrow{\scriptstyle z \mapsto z\overline{z}} & & \\ \mathbf{Z} & \xrightarrow{\mathrm{id}} & \mathbf{Z} & \subset & \mathbf{R}. \end{array}$$

12. Compute the structure of the group $\mathrm{Hom}(\mathbf{C}/\Lambda_1, \mathbf{C}/\Lambda_2)$ for each of the following choices of Λ_1 and Λ_2:
   a. $\Lambda_1 = \Lambda_2 = \mathbf{Z} + \mathbf{Z}\, i$;
   b. $\Lambda_1 = \mathbf{Z} + \mathbf{Z}\, i$ and $\Lambda_2 = \mathbf{Z} + \mathbf{Z}\, 2i$;
   b. $\Lambda_1 = \mathbf{Z} + \mathbf{Z}\, i$ and $\Lambda_2 = \mathbf{Z} + \mathbf{Z}\sqrt{-2}$.

13. Show that every group $\mathrm{Hom}(\mathbf{C}/\Lambda_1, \mathbf{C}/\Lambda_2)$ is a free abelian group of rank at most 2. Show that the rank is non-zero if $\mathbf{C}/\Lambda_1$ and $\mathbf{C}/\Lambda_2$ are isogenous, and that it is 2 if and only if $\mathbf{C}/\Lambda_1$ and $\mathbf{C}/\Lambda_2$ have complex multiplication by rings $\mathcal{O}_1$ and $\mathcal{O}_2$ having the same field of fractions.

14. A non-zero isogeny $\psi : T_1 \to T_2$ is said to be *cyclic* if $\ker \psi$ is a cyclic subgroup of T_1. Show that complex tori are isogenous if and only if there exists a cyclic isogeny between them. Show also that a torus admitting a cyclic endomorphism (different from the identity) has complex multiplication.

15. Show that the set $D \subset \mathcal{H}$ in 3.7 contains a *unique* representative of every $\mathrm{SL}_2(\mathbf{Z})$-orbit if we remove the elements on its boundary satisfying $|z| = 1$ and $\mathrm{Re}(z) > 0$.

*16. Let $f : E \to \widetilde{E}$ be a rational map between Weierstrass curves, i.e., a map of the form $(x, y) \mapsto (f_1(x, y), f_2(x, y))$ for functions $f_1, f_2 \in \mathcal{M}(E)$ with the property that the image of (x, y) lies in $\widetilde{E}(\mathbf{C})$ whenever it is defined. Show that f can be defined on all points of E, and that it corresponds to an analytic map of the corresponding tori.

17. Determine the Weierstrass polynomial $W(X)$ of the curve $\widetilde{E}$ in example 3.10 by proving the following statements.
    a. $W(X) = 4(X - \widetilde{\wp}(\frac{1}{2}\lambda_2))(X - \widetilde{\wp}(\frac{1}{4}\lambda_1))(X - \widetilde{\wp}(\frac{1}{4}\lambda_1 + \frac{1}{2}\lambda_2))$.
    b. We have $\widetilde{\wp}(\frac{1}{2}\lambda_2)) = -2a$.
    c. The function $4(\wp(z) - a)(\wp(z + \frac{1}{2}\lambda_1) - a)$ is constant with value $12a^2 - g_2$.
    d. We have $(\widetilde{\wp}(\frac{1}{4}\lambda_1) - a)^2 = 4(\wp(\frac{1}{4}\lambda_1) - a)^2 = 12a^2 - g_2$.
    e. $W(X) = 4(X + 2a)((X - a)^2 + g_2 - 12a^2) = 4(X + 2a)(X^2 - 2aX + g_2 - 11a^2)$.

18. Show that after a linear change of variables $X = 4(x - a)$ and $Y = 4y$, the equation of the Weierstrass curves E in 3.10 becomes $Y^2 = X(X^2 + \alpha X + \beta)$ with $\alpha = 12a$ and $\beta = 48a^2 - 4g_2$. Show that a similar change of variables then reduces the 2-isogenous curve to the form $Y^2 = X(X^2 - 2\alpha X + \alpha^2 - 4\beta)$.