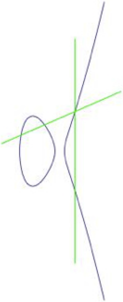
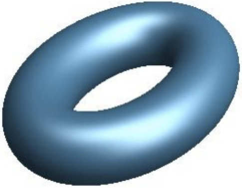
Elliptic curves with complex multiplication

Valerio Talamanca (Università Roma Tre & RNTA)





CIMPA research school on
Isogenies of elliptic curves and their applications to cryptography
Universidad del Cauca
July 24th - August 4th, 2023

Lattices in $\mathbb{C}$ and complex tori

Let $\Lambda \subset \mathbb{C}$ be a lattice and consider the associated complex tori defined as $\mathbb{C}/\Lambda$. Recall that the Weierstrass $\wp$-function defined as

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

gives rise, together with its derivative, to a map to the projective plane:

$$\begin{aligned} \Phi : \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}_2(\mathbb{C}) \\ z &\longmapsto [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1] \end{aligned}$$

whose image is an elliptic curve, that we will denote by E_Λ, which has Weierstrass equation

$$y^2 = 4x^3 + g_2(\Lambda)x + g_3(\Lambda)$$

The uniformization theorem tells us that every elliptic curve over $\mathbb{C}$ arises in this way.

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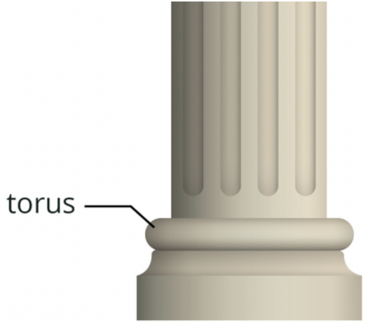

in architecture a torus is a particular roman moulding at the base of doric style column.

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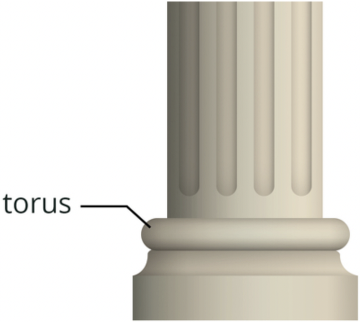

in architecture a torus is a particular roman moulding at the base of doric style column.
The latin word torus had several other meanings including rope, swelling, pillow, bed, coffin and lover.

Holomorphic maps of complex tori

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If α is such that $\alpha\Lambda_1 \subseteq \Lambda_2$, then we can define a surjective map $\phi_\alpha : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$, by setting

$$\phi_\alpha([z]_{\Lambda_1}) = [\alpha z]_{\Lambda_2}.$$

It can be shown that this is a holomorphic map.

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Consider

$$\{\alpha \in \mathbb{C} \,:\, \alpha\Lambda_1 \subseteq \Lambda_2\} \to \left\{ \frac{\mathbb{C}}{\Lambda_1} \xrightarrow{\phi} \frac{\mathbb{C}}{\Lambda_2} : \phi(0) = 0 \textit{ and } \phi \textit{ holomorphic} \right\}$$

$$\alpha \mapsto \phi_\alpha$$

Theorem

Let Λ_1 and Λ_2 be two lattices. Then the above association is a bijection. Moreover $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic if and only if Λ_1 and Λ_2 are homothetic.

Endomorphisms of elliptic curves over $\mathbb{C}$

In particular we get that given a lattice Λ and its associated elliptic curve E_Λ, the endomorphism ring of E_Λ is isomorphic to

$$\{\alpha \in \mathbb{C} \,:\, \alpha\Lambda \subset \Lambda\} = R_\Lambda$$

Note that we immediately recover the fact that $\mathrm{End}(E_\Lambda)$ contains $\mathbb{Z}$.

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In particular we get that given a lattice Λ and its associated elliptic curve E_Λ, the endomorphism ring of E_Λ is isomorphic to the following

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Note that we immediately recover the fact that $\mathrm{End}(E_\Lambda)$ contains $\mathbb{Z}$.
Moreover given $\alpha \in R_\Lambda$ we fix $[\alpha] : E_\Lambda \to E_\Lambda$, by requiring that the following diagram is commutative

$$\begin{tikzcd}
\mathbb{C}/\Lambda \arrow[r, "\phi_\alpha"] \arrow[d, "\Phi"] & \mathbb{C}/\Lambda \arrow[d, "\Phi"] \\
E_\Lambda \arrow[r, "{[\alpha]}"] & E_\Lambda
\end{tikzcd}$$

Complex multiplication

Definition

Let $E/\mathbb{C}$ be an elliptic curve. We say that E as complex multiplication (CM for short) if $\mathrm{End}(E)$ *is strictly bigger than* $\mathbb{Z}$.

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Example

Let E be the elliptic curve $y^2 = x^3 + x$, then the map $(x, y) \mapsto (-x, iy)$ induces an endomorphism ϕ of E. Clearly ϕ has order 4, *and so* $\mathrm{End}(E)$ *is bigger than* $\mathbb{Z}$ *and hence E has complex multiplication.*

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Let E be the elliptic curve having Weierstrass equation $y^2 = x^3 + x$, then the map $(x, y) \mapsto (-x, iy)$ induces an endomorphism ϕ of E. Clearly ϕ has order 4, *and so* $\mathrm{End}(E)$ *is bigger than $\mathbb{Z}$ and hence E has complex multiplication.*

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Let E be the elliptic curve having Weierstrass equation $y^2 = x^3 + 1$, and let ρ be a primitive cubic root of unity. Then the map $(x, y) \mapsto (\rho x, y)$ induces an endomorphism ϕ of E. Clearly ϕ has order 3, *and so E has complex multiplication.*

Complex multiplication

Definition

Let $E/\mathbb{C}$ be an elliptic curve. We say that E as complex multiplication (CM for short) if $\mathrm{End}(E)$ *is strictly bigger than $\mathbb{Z}$.*

So if E is a complex CM elliptic curve, then $\mathrm{End}(E) \otimes \mathbb{Q}$ is a quadratic imaginary field, and $\mathrm{End}(E)$ is an order in that field.
As a matter of notation if $\mathrm{End}(E) \cong \mathcal{O} \subset \mathbb{C}$ and $K = \mathcal{O} \otimes \mathbb{Q}$ we will say that E has complex multiplication by $\mathcal{O}$, or that E has complex multiplication by K.

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But we do not miss much doing so as the next theorem shows:

Theorem

Suppose that E has complex multiplication by an order $\mathcal{O} \subset K$. Then there exists an elliptic curve E' isogenous to E and having complex multiplication by $\mathcal{O}_K$.

Construction of elliptic curves with complex multiplication

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Suppose $\mathfrak{a} \subset \mathcal{O}_K$ is a fractional ideal. Then $\mathfrak{a} \subset K \subset \mathbb{C}$ is a lattice in $\mathbb{C}$. Consider $E_\mathfrak{a}$, then its endomorphism ring is given by

$$\begin{aligned} \operatorname{End}(E_\mathfrak{a}) &\cong \{\alpha \in \mathbb{C} : \alpha\mathfrak{a} \subset \mathfrak{a}\} \\ &= \{\alpha \in K, : \alpha\mathfrak{a} \subset \mathfrak{a}\} \\ {}^1 &= \mathcal{O}_K \end{aligned}$$

[1] Exercise: prove this equality

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Recall that given two lattices Λ_1 and Λ_2, then E_{Λ_1} and E_{Λ_2} are isomorphic if and only if Λ_1 and Λ_2 are homothetic.

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Recall that given two lattices Λ_1 and Λ_2, then E_{Λ_1} and E_{Λ_2} are isomorphic if and only if Λ_1 and Λ_2 are homothetic.

Let $\overline{\mathfrak{a}}$ denote the class of $\mathfrak{a}$ in $Cl(\mathcal{O}_K)$, the class group of $\mathcal{O}_K$. Thus if $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ (i.e. there exists $c \in K$ such that $c\mathfrak{a} = \mathfrak{b}$) then $E_{\mathfrak{a}}$ and $E_{\mathfrak{b}}$ are isomorphic. It follows that we have a map

$$Cl(\mathcal{O}_K) \to \mathrm{Ell}(\mathcal{O}_K)$$
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Thus if $\mathfrak{a}$ and $\mathfrak{b}$ are homothetic (i.e. $c\mathfrak{a} = \mathfrak{b}$) then $E_{\mathfrak{a}}$ and $E_{\mathfrak{b}}$ are isomorphic. It follows that we have a map

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Moreover it is injective: $E_{\mathfrak{a}} \cong E_{\mathfrak{b}} \iff$ there exists $c \in \mathbb{C}$ such that $\mathfrak{a} = c\mathfrak{b}$. But then $c \in K$ and so $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$.

Let Λ be a lattice such that E_Λ has complex multiplication by $\mathcal{O}_K$. For any $\mathfrak{a}$, we set:

$$\mathfrak{a}\Lambda = \{\alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r \ : \ \alpha_i \in \mathfrak{a}, \lambda_i \in \Lambda\}$$

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Proposition

Let $\Lambda \subset \mathbb{C}$ be a lattice. Assume that E_Λ has complex multiplication by $\mathcal{O}_K$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be non zero fractional ideal of K. Then

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- $\mathfrak{a}\Lambda$ *is a lattice in* $\mathbb{C}$
- $E_{\mathfrak{a}\Lambda}$ *has complex multiplication by* $\mathcal{O}_K$
- $E_{\mathfrak{a}\Lambda} \cong E_{\mathfrak{b}\Lambda}$ *if and only if* $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ *in* $Cl(\mathcal{O}_K)$.

Proposition

Let K be an imaginary quadratic number field, and $\mathcal{O}_K$ its ring of integers. The map

$$Cl(\mathcal{O}_K) \times \mathrm{Ell}(\mathcal{O}_K) \to \mathrm{Ell}(\mathcal{O}_K)$$
$$(\bar{\mathfrak{a}}, E_\Lambda) \mapsto \bar{\mathfrak{a}} * E_\Lambda = E_{\mathfrak{a}^{-1}\Lambda}$$

is a simply transitive action of $Cl(\mathcal{O}_K)$ on $\mathrm{Ell}(\mathcal{O}_K)$. In particular

$$\# Cl(\mathcal{O}_K) = \# \mathrm{Ell}(\mathcal{O}_K)$$

$\mathfrak{a}$-torsion points

Let E be an elliptic curve with complex multiplication by K. For any ideal $\mathfrak{a}$ of $\mathcal{O}_K$ we set

$$E[\mathfrak{a}] = \{P \in E \,:\, [\alpha]P = 0 \text{ for all } \alpha \in \mathfrak{a}\}$$

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Suppose that $E = E_\Lambda$, as usual Λ a lattice in $\mathbb{C}$, and fix an isomorphism $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$. Since $\Lambda \subset \mathfrak{a}^{-1}\Lambda$ we have a natural isogeny $\mathbb{C}/\Lambda \to \mathbb{C}/\mathfrak{a}^{-1}\Lambda$ and hence a natural isogeny $E \to \overline{\mathfrak{a}} * E = E_{\mathfrak{a}^{-1}\Lambda}$.

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Let's first look on the complex tori side. Suppose that $E = E_\Lambda$, as usual Λ a lattice in $\mathbb{C}$. Then

$$\mathbb{C}/\Lambda[\mathfrak{a}] = \{z \in \mathbb{C} : \alpha z = 0 \text{ for all } \alpha \in \mathfrak{a}\}$$

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&= \{z \in \mathbb{C} \,:\, \alpha z \in \Lambda \text{ for all } \alpha \in \mathfrak{a}\,\}/\Lambda \\
&= \{z \in \mathbb{C} \,:\, z\mathfrak{a} \in \Lambda\}/\Lambda \\
&= \mathfrak{a}^{-1}\Lambda/\Lambda \\
&= \ker\left(\mathbb{C}/\Lambda \xrightarrow{z \mapsto z} \mathbb{C}/\mathfrak{a}^{-1}\Lambda\right)
\end{aligned}$$

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So we have $(\mathbb{C}/\Lambda)[\mathfrak{a}] = \ker\left(\mathbb{C}/\Lambda \xrightarrow{z \mapsto z} \mathbb{C}/\mathfrak{a}^{-1}\Lambda\right)$. Fix an analytic isomorphism from $(\mathbb{C}/\Lambda)$ to $E(\mathbb{C})$. Then $(\mathbb{C}/\Lambda)[\mathfrak{a}]$ corresponds to $E[\mathfrak{a}]$ and $\ker\left(\mathbb{C}/\Lambda \xrightarrow{z \mapsto z} \mathbb{C}/\mathfrak{a}^{-1}\Lambda\right)$ corresponds to the kernel of the natural isogeny from E to $\overline{\mathfrak{a}} * E$.

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Corollary

Let E be an elliptic curve with complex multiplication by $\mathcal{O}_K$.

- *Let $\mathfrak{a}$ be an integral ideal, then the natural isogeny $E \to \overline{\mathfrak{a}} * E$ has degree $\mathrm{N}_{\mathbb{Q}}^{K}(\mathfrak{a})$.*

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- *Let $\alpha \in \mathcal{O}_K$, then the endomorphism $[\alpha] \in \mathrm{End}(E)$ has degree $|\mathrm{N}_{\mathbb{Q}}^{K}(\alpha)|$.*

From $\mathbb{C}$ to $\overline{\mathbb{Q}}$

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You can freely use the following fact:

Fact

Let $\alpha \in \mathbb{C}$ be such that the set $\{\sigma(\alpha) : \sigma \in \mathrm{Aut}(\mathbb{C})\}$ is finite, the α is an algebraic number.

From $\mathbb{C}$ to $\overline{\mathbb{Q}}$

If F is any field set

$$\mathrm{Ell}_F(\mathcal{O}_K) = \frac{\{\text{Elliptic curves } E/F \text{ with } \mathrm{End}(E) \cong \mathcal{O}_K\}}{\text{isomorphism over } F}$$

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then one has that ι is a bijection. To prove it we need to recall the following result about elliptic curves:

Theorem

Two elliptic curves E and E' over an algebraically closed field $\overline{L}$ are isomorphic if and only they have the same j-invariant. Moreover if $j_0 \in \overline{L}$, then there exists an elliptic curve E_0 defined over $L(j_0)$ such that $j(E_0) = j_0$.

Consider $\mathrm{Ell}_{\overline{\mathbb{Q}}}(\mathcal{O}_K)$. On it we have an action of $\mathrm{Gal}(\overline{K}/K)$, sending E to E^σ. Recall that we have a transitive action of $Cl(\mathcal{O}_K)$ so it must exists $\overline{\mathfrak{a}} \in Cl(\mathcal{O}_K)$ such that

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Now the amazing fact is that actually $\mathfrak{a}_E$ does not depends on E.

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Theorem

Let $K/\mathbb{Q}$ be an imaginary quadratic field. Then there exists a homomorphism $\Psi : \mathrm{Gal}(\overline{K}/K) \to Cl(\mathcal{O}_K)$, uniquely determined by requiring that $E^\sigma = \Psi(\sigma) * E$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and all $E \in \mathrm{Ell}_{\overline{\mathbb{Q}}}(\mathcal{O}_K)$.

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- *Set* $\# Cl(\mathcal{O}_K) = h_k$ *and suppose that* $E_1, \ldots E_{h_k}$ *be a complete set of representatives for* $\operatorname{Ell}(\mathcal{O}_K)$. *Then* $j(E_1), \ldots j(E_{h_k})$, *is a complete set of* $\operatorname{Gal}(\overline{K}/K)$ *conjugates for* $j(E)$

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- *For every non zero fractional ideal $\mathfrak{a}$ of K we have:*

$$j(E)^{[\mathfrak{a}, H/K]} = j(\overline{\mathfrak{a}} * E)$$

where $[\mathfrak{a}, H/K] \in \mathrm{Gal}(H/K)$ is the Artin symbol of $\mathfrak{a}$.