Explicit Methods in Algebraic Number Theory

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1 Lecture 3

1.1 Ideal Class Group

Let J_K the group of fractional ideals of K and P_K the subgroup of J_K of principal fractional ideals. Class group of K is the quotient $Cl(K) = J_K/P_K$. It is known that it is a finite abelian group and its order h_K is called the class number of K. In general, $\mathcal{O}_K$ is no a UFD. However, $Cl(K)$ is trivial if and only if $\mathcal{O}_K$ is a PID, which is equivalent to be an UFD. Let us recall that, the main steps to prove the finiteness of $Cl(K)$ are the following:

- Every ideal class contains an integral ideal $\mathfrak{a}$ such that
$$N(\mathfrak{a}) \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2}\sqrt{|D_K|} \quad \text{(Minkowski bound)},$$
where $n = r_1 + 2r_2$
- There are finitely many ideals $\mathfrak{a}$ with $N(\mathfrak{a})$ bounded.

Theorem 1.1. *Let $\mathfrak{a}$ be a fractional ideal in K. Then $\mathfrak{a}$ may be written in a unique way (up order) by*
$$\mathfrak{a} = \prod_{i=1}^{r} \mathfrak{p}_i^{v_i},$$
where $v_i \in \mathbb{Z}$.

Example 1. *Find h_k if $K = \mathbb{Q}(\sqrt{-14})$.*
Minkowski bound: $\frac{2!}{2^2}\left(\frac{4}{\pi}\right)^0\sqrt{4\cdot 14} = \sqrt{14} < 4$. Then, every class ideal has an integral representative $\mathfrak{a}$ with $N(\mathfrak{a}) < 4$. Note that if $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2\ldots\mathfrak{p}_r$, then $N(\mathfrak{p}) = p^f$, so $p = 2$ or 3. Let us see the factorization of $2\mathcal{O}_K$ and $3\mathcal{O}_K$.

- *$x^2 - 14 \equiv x^2$ (mod 2), then $2\mathcal{O}_K = (2, \sqrt{14})^2$.*
- *$x^2 - 14 \equiv x^2 + 1$ (mod 3) which is irreducible, then $2\mathcal{O}_K$ is prime.*

Therefore, $\mathfrak{p} = (2, \sqrt{14})$ or $(3).(2, \sqrt{14})$ is a principal ideal if and only if there is an element $a + b\sqrt{14}$ with $N(a + b\sqrt{14}) = \pm 2$ and $(2, \sqrt{14}) = (a + b\sqrt{14})$. It is no difficult to prove that $(2, \sqrt{14}) = (4 + \sqrt{14})$ so $h_K = 1$.

1.2 Analytic Class Number Formula

If K is a number field, we define Dedekind zeta function associated to K by

$$\zeta_K(s) = \sum_{\mathfrak{a}\neq 0 \text{ ideal}} \frac{1}{N(\mathfrak{a})^s}, \ s \in \mathbb{C}.$$

This function encodes a lot of information about the number field K. Properties:

- $\zeta_K(s) = \prod_{\mathfrak{p} \text{ prime}}(1 - N(\mathfrak{p})^{-s})^{-1}$, is an holomorphic function if Re(s)¿1.
- In $s = 1$, $\zeta_K(s)$ is divergent.
- It has an meromorphic continuation to the left side of Re(s)¿1.

Theorem 1.2. *Let K be a number field of degree n over $\mathbb{Q}$ with r_1 and r_2 the number of real and nonreal embedding over $\mathbb{C}$. Then $\zeta_K(s)$ extends to a meromorphic function defined for all $s \in \mathbb{C}$ with a simple pole in $s = 1$ and*

$$\lim_{s\to\infty} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{|W_K|\sqrt{|D_K|}},$$

where W_K is the group of unity in $\mathcal{O}_K$ and R_K is the regulator of K.

1.3 Class Number of Quadratic Number Fields

If K is a quadratic numbe field, let us consider the Dirichlet function associated to the quadratic character given by the extended Jacobi symbol $\chi_K(m) = \left(\frac{D_K}{m}\right)$,

$$L(\chi_K, s) = \sum_{n=1}^{\infty} \frac{\chi_K(m)}{n^s}.$$

This function, is related with the class number by the following identity:

$$\lim_{s\to\infty} \zeta_K(s) = L(\chi_K, 1).$$