Limiting values of Lambert series and the secant zeta-function

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Abstract

We extract some topics from some recent papers as well as from [KT] and show merging of analytic number theory and the theory of modular forms. The first topic is about one of Davenport-Chowla identity (1) in which there appear the Liouville function $\lambda(n)$ which is one of the most important ingredients in PNT as well as the Riemann's everywhere non-differentiable function.

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n}\psi(nx) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi n^2 x}{n^2}. \tag{1}$$

One of main objectives of the talk is to exhibit the Abel-Tauber process, where Abelian process means certain integration and Tauberian means certain differentiation. To establish (1), one first proves the integrated equality, which follows the functional equation–zeta symmetry–only. To prove (1) from this, one differentiates termwise and for this uniformity of convergence needs to be established. For this, deeper estimates of the sum of Liouville function is needed, an equivalent to the PNT. Thus this is an example of the threshold phenomenon, i.e. to what extent one can obtain from the zeta-symmetry and how much is needed the zero-free region which comes from the Euler product.

Then we move on to limiting values of modular functions. In Riemann's fragment [Rie2], there already appears the notion of radial limit after

integrating along a radius. In the upper half-plane this corresponds to the integral along a vertical line.

In [CKL] we applies the theta-transformation formula, which is equivalent to the functional equation, to derive differentiablility (at certain rational points) of the Riemann function as well as the Gauss quadratic reciprocity law. To deduce the former one need integral along a line segment parallel to the real axis while deducing the latter one needs integration along a slanted line segment.

Recently, Lalín et al. [LRR] considered the secant zeta function

$$\psi(z,s)=\sum_{n=1}^{\infty}\frac{\sec(n\pi z)}{n^s}. \tag{2}$$

Their main result Lalín et al [LRR, Theorem 3] is that the difference

$$(\alpha+1)^{l-1}\psi\left(\frac{\alpha}{\alpha+1},l\right)-(-\alpha+1)^{l-1}\psi\left(\frac{\alpha}{-\alpha+1},l\right) \tag{3}$$

can be expressed in terms of Bernoulli and Euler numbers. In [KKM] we have located this as a special case of the following

Corollary 1.

$$\begin{aligned}(\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1},s,\left(\frac{1}{2},0\right)\right)+(\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1},s,\left(\frac{1}{2},0\right)\right)\\ =\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{\left(1-e\left[\frac{s}{2}\right]\right)}\int_{I(\lambda,\infty)}t^{s-1}\sum_{m=0}^{\infty}2^{-m-1}E_m\frac{t^m}{m!}\sum_{n=0}^{\infty}(2^{1-n}-1)B_n\\ \times\frac{\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\}t^{n-1}}{n!}\,\mathrm{d}t.\end{aligned} \tag{4}$$

References

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