Compatible systems, day 2

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Escuela CIMPA 2025

PART 1: Canonical representation

The maps ϵ and α_ℓ

Let K be a number field, m a modulus. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times/O_K^\times \cap U_m & \longrightarrow & \mathbb{I}_K/U_m & \longrightarrow & C_m & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_m(\mathbb{Q}) & \longrightarrow & S_m(\mathbb{Q}) & \longrightarrow & C_m & \longrightarrow & 1 \end{array}$$

We'll use the maps $\epsilon : \mathbb{I}_K \to \mathbb{I}_K/U_m \to S_m(\mathbb{Q})$ and

$$\alpha_\ell : \mathbb{I}_K \to \prod_{v|\ell} K_v^\times = (K \otimes \mathbb{Q}_\ell)^\times = T(\mathbb{Q}_\ell) \to T_m(\mathbb{Q}_\ell) \to S_m(\mathbb{Q}_\ell).$$

From the diagram, ϵ y α_ℓ agree on $K^\times \subseteq \mathbb{I}_K$.

The map ϵ_ℓ

We have $\epsilon : \mathbb{I}_K \to S_m(\mathbb{Q})$ and $\alpha_\ell : \mathbb{I}_K \to S_m(\mathbb{Q}_\ell)$ that agree on $K^\times$. Thus,

$$\epsilon(x)\alpha_\ell(x)^{-1}$$

is the neutral element of $S_m(\mathbb{Q}_\ell)$ for all $x \in K^\times$. So we can define

$$\epsilon_\ell = \epsilon \cdot \alpha_\ell^{-1} : C_K = \mathbb{I}_K / K^\times \to S_m(\mathbb{Q}_\ell)$$

on the idele class group.

Canonical ℓ-adic representation of G^{ab}

Write $G^{ab} = \mathrm{Gal}(K^{ab}/K)$. Let D_K be the identity component of C_K. Artin's isomorphism from Class Field Theory is

$$C_K/D_K \simeq G^{ab}.$$

The ℓ-adic Lie group $S_m(\mathbb{Q}_\ell)$is totally disconnected, hence

$$\epsilon_\ell(D_K) = 1 \in S_m(\mathbb{Q}_\ell).$$

We get the **canonical ℓ-adic representation** valued on S_m (not on $GL(V)$):

$$\epsilon_\ell : G^{ab} \simeq C_K/D_K \to S_m(\mathbb{Q}_\ell).$$

This is unramified away from m.

Imagen de Frobenius: sistemas compatibles

For $v \nmid m$ we define $\mathrm{Frob}_v \in S_m(\mathbb{Q})$ as follows:
Choose $f_v \in \mathbb{I}_K$ a uniformizer at v, and 1 on the other components.
Define $\mathrm{Frob}_v = \epsilon(f_v) \in S_m(\mathbb{Q})$.

Class field theory shows that $f_v \in \mathbb{I}_K$ gives the Frobenius class modulo inertia in G^{ab}. Hence $\epsilon_\ell(F_v) \in S_m(\mathbb{Q}_\ell)$ is well-defined when $v \nmid \ell m$.

Theorem

The elements $\mathrm{Frob}_v \in S_m(\mathbb{Q})$ are well-defined.
If $v \nmid \ell m$ then $\epsilon_\ell(F_v) = \mathrm{Frob}_v \in S_m(\mathbb{Q})$, which is independent of ℓ.

The second part holds because $\alpha_\ell(f_v) = 1$ para $v \nmid \ell m$. It implies the first.

In this sense, $\{\epsilon_\ell : G^{ab} \to S_m(\mathbb{Q}_\ell)\}_\ell$ **is a strictly compatible rational system** whose ramification is contained in m.

Summary

- $\epsilon : \mathbb{I}_K \to S_m(\mathbb{Q})$
- $\alpha_\ell : \mathbb{I}_K \to S_m(\mathbb{Q}_\ell)$
- $\epsilon_\ell = \epsilon \cdot \alpha_\ell^{-1} : C_K = \mathbb{I}_K / K^\times \to S_m(\mathbb{Q}_\ell)$
- By class field theory $G^{ab} = C_K / D_K$ so we get the canonical representations with values on S_m:

  $$\epsilon_\ell : G^{ab} \to S_m(\mathbb{Q}_\ell)$$

  unramified away from ℓm.
- $\epsilon_\ell(F_v) = \mathrm{Frob}_v \in S_m(\mathbb{Q})$ are independent of ℓ.
- We get a system of ℓ-adic representations which is rational and strictly compatible, valued on S_m, not $GL(V)$.

PART 2: Characters

Character group

Let k be a field of characteristic 0 and H and affine commutative algebraic group defined over k. It **character group** is

$$X(H) = \mathrm{Hom}(H_{k^{\mathrm{alg}}}, \mathbb{G}_{m,k^{\mathrm{alg}}})$$

The Galois group $G_k = \mathrm{Gal}(k^{\mathrm{alg}}/k)$ acts on $X(H)$.
We use this with $H = S_m$.

Trace

Given V a k-vector space of finite dimension we have the algebraic group $GL(V)$. For a semisimple representation

$$\phi : H \to GL(V)$$

its trace is

$$\theta_\phi = \sum_\chi n_\chi(\phi)\chi \in \mathbb{Z}[X(H)]$$

according to the decomposition of ϕ into characters.
For S_m we have for free that ϕ is semisimple (S_m is of multiplicative type)

Classification

Let $Rep_k(H)$ be the set of isomorphism classes of semisimple representations of H. The next result is a version of Maschke's theorem.

Theorem

The trace $\phi \mapsto \theta_\phi$ defines a bijection between $Rep_k(H)$ and formal sums $\theta \in \mathbb{Z}[X(H)]$ satisfying:

- *θ is Galois invariant (i.e. for G_k acting on the $\chi \in X(H)$), and*
- *the coefficients of θ are non-negative.*

Field of definition

Let k'/k be a field extension of characteristic 0.

Theorem

Let $\phi \in Rep_{k'}(S_{m,k'})$ that is $\phi : S_{m,k'} \to GL(V)$. TFAE:

(1) *ϕ comes from $Rep_k(S_{m,k})$, and*

(2) *For each $v \nmid m$, the characteristic polynomial of $\phi(\mathrm{Frob}_v)$ is in $k[T]$.*

(1) $\to$ (2) is clear, and the reciprocal comes from Chebotarev and the fact that the maps $\epsilon_\ell : G^{ab} \to S_m(\mathbb{Q}_\ell)$ have Zariski dense image.

PART III: Compatible systems

ℓ-adic case

Back to a number field K and modulus m.
Let V_ℓ be a $\mathbb{Q}_\ell$-vector space of finite dimension and

$$\phi : S_{m,\mathbb{Q}_\ell} \to GL(V_\ell)$$

be a representation of algebraic groups. We get the continuous map

$$\phi : S_m(\mathbb{Q}_\ell) \to GL(V_\ell).$$

Using $\epsilon_\ell : G^{ab} \to S_m(\mathbb{Q}_\ell)$ we obtain an ℓ-adic representation

$$\phi_\ell = \phi \circ \epsilon_\ell : G^{ab} \to GL(V_\ell).$$

ℓ-adic case

Theorem

Properties of $\phi_\ell : G^{ab} \to GL(V_\ell)$:

(i) *It is semisimple*

(ii) *Unramified for each* $v \nmid \ell m$

(iii) $\phi_\ell(F_v) = \phi(\mathrm{Frob}_v)$ *(recall:* $\mathrm{Frob}_v \in S_m(\mathbb{Q})$ *is independent of* ℓ*)*

(iv) ϕ_ℓ *is rational if and only if* ϕ *comes from* $Rep_{\mathbb{Q}}(S_m)$.

(i): S_m is of multiplicative type.
(ii), (iii): from the theory of the canonical representations of S_m.
(iv): Last theorem of PART II from today.

Global case

We want to "glue" the different ϕ_ℓ as ℓ varies, obtaining a compatible system. This is possible if we start with a $\mathbb{Q}$-vector space V_0 and a representation

$$\phi_0 : S_m \to GL(V_0)$$

defined over $\mathbb{Q}$. Although we dont have "$G^{ab} \to S_m(\mathbb{Q})$" we still can extend scalars to $\mathbb{Q}_\ell$ and then reduce to the ℓ-adic case using ϵ_ℓ.
We get

$$\phi_\ell : G^{ab} \to GL(V_{0,\ell}), \quad V_{o,\ell} = V_0 \otimes \mathbb{Q}_\ell.$$

Sistemas compatibles

Here is the construction of $\phi_\ell : G^{ab} \to GL(V_{0,\ell})$ as ℓ varies:

$$\begin{array}{ccccc} & & S_m(\mathbb{Q}) & \xrightarrow{\phi_0} & GL(V_0) \\ & & \downarrow & & \downarrow \\ G^{ab} & \xrightarrow{\epsilon_\ell} & S_m(\mathbb{Q}_\ell) & \xrightarrow{\phi_{0,\ell}} & GL(V_{0,\ell}) \end{array}$$

Theorem

Properties of the system $\{\phi_\ell : G^{ab} \to GL(V_{0,\ell})\}_\ell$:

(1) *It is rational, strictly compatible, semisimple.*

(2) *Unramified away from m, and $\phi_{0,\ell}(F_v) = \phi_0(\mathrm{Frob}_v)$ for each $v \nmid \ell m$*

(3) *there are infinitely many ℓ such that ϕ_ℓ is diagonalizable.*

The last part is by (the refined version of) Chebotarev.

PART IV: Introduction to abelian locally algebraic representations.

The plan

- Define abelian **locally algebraic** representations. Locally, they come from a morphism of algebraic groups $T \to GL(V)$.
- We'll se that, if rational, they come from our previous contructions using S_m (recall: $T \to T_m \to S_m$)
- We'll discuss a criterion for being locally algebraic based on the existence of a Hodge–Tate decomposition.

Today: just the first item.

Local case

Let ℓ be a prime and L a finite extension of $\mathbb{Q}_\ell$. Consider the torus

$$T = Res_{L/\mathbb{Q}_\ell}\mathbb{G}_m.$$

Let $\rho : G_L^{ab} = \mathrm{Gal}(L^{ab}/L) \to GL(V)$ be an abelian ℓ-adic representation. By local class field theory we get a map

$$i : L^\times \to G_L^{ab}$$

obtaining a map

$$\rho \circ i : L^\times = T(\mathbb{Q}_\ell) \to GL(V).$$

Definition. ρ is **locally algebraic** if there is a morphism of algebraic groups $r : T \to GL(V)$ such that near $1 \in L^\times$ we have $(\rho \circ i)(x) = r(x^{-1})$.

Remark. Such an r is unique.

Global case

Back to the number field case.
Let ℓ be a prime and

$$\rho : G^{ab} \to GL(V_\ell)$$

an abelian ℓ-adic representation. Let $v|\ell$ be a place of K with decomposition group $D_v \subseteq G^{ab}$. We have $D_v \simeq G_L^{ab}$ with $L = K_v$. The we get

$$\rho_v : G_L^{ab} \to D_v \subseteq G^{ab} \to GL(V)$$

Definition. ρ is locally algebraic if for each such $v|\ell$ the local representation ρ_v is locally algebraic.

Remark. Again, the corresponding maps r are unique.

Equivalent definition

By the component embedding and class field theory we have the map

$$i_\ell : K_\ell^\times = \prod_{v|\ell} K_v^\times \to \mathbb{I}_k \to C_K \to G^{ab}.$$

Also, note that $T(\mathbb{Q}_\ell) = (K \otimes \mathbb{Q}_\ell)^\times = K_\ell^\times = \prod_{v|\ell} K_v^\times$.

Theorem

$\rho : G^{ab} \to GL(V_\ell)$ is locally algebraic if and only if there is a map of algebraic groups over $\mathbb{Q}_\ell$

$$f : T_{\mathbb{Q}_\ell} \to GL(V_\ell)$$

such that $\rho \circ i_\ell(x) = f(x^{-1}) = f(x)^{-1}$ for all $x \in K_\ell^\times$ close enough to 1.

Remark. The f is unique.

End of day 2