Compatible systems, day 3

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PART 1: Locally algebraic representations, local case

Definition in the local case

Let ℓ be a prime and L a finite extension of $\mathbb{Q}_\ell$. Consider the torus

$$T = Res_{L/\mathbb{Q}_\ell}\mathbb{G}_m.$$

Let $\rho : G_L^{ab} = \mathrm{Gal}(L^{ab}/L) \to GL(V)$ be an abelian ℓ-adic representation. By local class field theory we get a map

$$i : L^\times \to G_L^{ab}$$

obtaining a map

$$\rho \circ i : L^\times = T(\mathbb{Q}_\ell) \to GL(V).$$

Definition. ρ is **locally algebraic** if there is a morphism of algebraic groups $r : T \to GL(V)$ such that near $1 \in L^\times$ we have $(\rho \circ i)(x) = r(x^{-1})$.

Remark. Such an r is unique.

Example

Let's work out the simplest case:

Let $L = \mathbb{Q}_\ell$ and let V be a 1-dimensional $\mathbb{Q}_\ell$-vector space. Then ρ is

$$\rho : G_L^{ab} \to GL(V) = \mathbb{Q}_\ell^\times.$$

The local Artin map is $i : \mathbb{Q}_\ell^\times \to G_L^{ab}$ so we get a continuous group morphism

$$\rho \circ i : \mathbb{Q}_\ell^\times \to \mathbb{Q}_\ell^\times.$$

Using the theory of ℓ-adic Lie groups (concretely, the logarithm map) we see that near 1 the map is given by $x \mapsto x^\gamma$ for some $\gamma \in \mathbb{Z}_\ell$.

Thus, ρ is locally algebraic exactly when $\gamma \in \mathbb{Z}$, in which case $r(x) = x^{-\gamma}$.

A test for being locally algebraic

Recall that L is a finite extension of $\mathbb{Q}_\ell$. We aim to give an easy-to-check test for an abelian ℓ-adic representation $\rho : G_L^{ab} \to GL(V)$ to be locally algebraic. The result is

Theorem (Tate)

Let $\rho : G_L^{ab} \to GL(V)$ be an abelian ℓ-adic representation. TFAE

- *ρ is locally algebraic*
- *ρ is of Hodge–Tate type and its restriction to the inertia subgroup $I \leq G_L^{ab}$ is semisimple.*

The proof is too technical for us; it is in the Appendix of Ch. III of Serre's book. We will be happy if we can make sense of the statement.

The condition on inertia should be understandable (and you studied it for elliptic curves); we will explain the "Hodge–Tate type" part.

The subspaces W^i

L is a finite extension of $\mathbb{Q}_\ell$ and $\mathbb{C}_\ell$ is the completion of L^{alg}.
$G_L = \mathrm{Gal}(L^{\mathrm{alg}}/L)$ acts continuously on $\mathbb{C}_\ell$.

Let W be a $\mathbb{C}_\ell$-vector space of finite dimension on which G_L acts continuously and semilinearly:

$$\sigma(c \cdot v) = \sigma(c) \cdot \sigma(v).$$

Let $\chi : G_L \to \mathbb{Z}_\ell^\times$ be the cyclotomic character that we constructed from roots of 1.
For $i \in \mathbb{Z}$ we define the χ^i-**isotypical component** of W:

$$W^i = \{w \in W : \forall \sigma \in G_L, \quad \sigma(w) = \chi(\sigma)^i \cdot w\} \subseteq W.$$

This is an L-vector subspace of W because G_L acts trivially on L. Not a $\mathbb{C}_\ell$-vector subspace in general.

The spaces $W(i)$

Since W^i are not $\mathbb{C}_\ell$-vector subspace, we cannot hope to directly use them to decompose W. There is a solution to this problem.

Let $W(i) = \mathbb{C}_\ell \otimes_L W^i$. Then this is a $\mathbb{C}_\ell$-vector space and G_L acts on it:

$$\sigma(c \otimes w) := \sigma(c) \otimes \sigma(w) = \sigma(c) \otimes (\chi(\sigma)^i w) = \chi(\sigma)^i \sigma(c) \otimes w \in W(i).$$

We get G_L-equivariant $\mathbb{C}_\ell$-linear maps $a_i : W(i) \to W$.

Theorem (Tate)

The map

$$a : \bigoplus_{i \in \mathbb{Z}} W(i) \to W$$

induced by the a_i, is injective.

In particular, W^i have finite L-dimension and only finitely many of them are non-zero.

Hodge–Tate type

Definition. We say that W is of **Hodge–Tate type** if

$$a : \bigoplus_i W(i) \to W$$

is an isomorphism (only need to check surjectivity).

Now let $\rho : G_L \to GL(V)$ be an ℓ-adic representation.

Definition. ρ is of **Hodge–Tate type** if $W := \mathbb{C}_\ell \otimes_L V$ is.

The result again

Recall: L is a finite extension of $\mathbb{Q}_\ell$.

Theorem (Tate)

Let $\rho : G_L^{ab} \to GL(V)$ be an abelian ℓ-adic representation. TFAE

- *ρ is locally algebraic*
- *ρ is of Hodge–Tate type and its restriction to the inertia subgroup $I \leq G_L^{ab}$ is semisimple.*

Now it should make sense to you.

PART 2: Locally algebraic representations, global case

Global case

Back to the number field case K.
Let ℓ be a prime and

$$\rho : G^{ab} \to GL(V_\ell)$$

an abelian ℓ-adic representation. Let $v|\ell$ be a place of K with decomposition group $D_v \subseteq G^{ab}$. We have $D_v \simeq G_L^{ab}$ with $L = K_v$. Then we get

$$\rho_v : G_L^{ab} \to D_v \subseteq G^{ab} \to GL(V)$$

Definition. ρ is locally algebraic if for each such $v|\ell$ the local representation ρ_v is locally algebraic.

Remark. Again, the corresponding maps r are unique.

Equivalent definition

By the component embedding and class field theory we have the map

$$i_\ell : K_\ell^\times = \prod_{v|\ell} K_v^\times \to \mathbb{I}_k \to C_K \to G^{ab}.$$

Also, note that $T(\mathbb{Q}_\ell) = (K \otimes \mathbb{Q}_\ell)^\times = K_\ell^\times = \prod_{v|\ell} K_v^\times$.

Theorem

$\rho : G^{ab} \to GL(V_\ell)$ is locally algebraic if and only if there is a map of algebraic groups over $\mathbb{Q}_\ell$

$$f : T_{\mathbb{Q}_\ell} \to GL(V_\ell)$$

such that $\rho \circ i_\ell(x) = f(x^{-1}) = f(x)^{-1}$ for all $x \in K_\ell^\times$ close enough to 1.

Remark. The f is unique.

Modulus of a locally algebraic abelian representation

Let $\rho : G^{ab} \to GL(V_\ell)$ be locally algebraic with associated (algebraic) morphism

$$f : T_{\mathbb{Q}_\ell} \to GL(V_\ell).$$

Recall the Artin map $i : \mathbb{I}_K \to G^{ab}$ and, for a modulus m, the subgroups

$$U_{v,m} = 1 + \mathfrak{p}_v^{m(v)} \subseteq K_v^\times.$$

Note that $\prod_{v|\ell} U_{v,m} \subseteq K_\ell^\times = T_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$.

Definition. m is a modulus of definition for ρ if

- $\rho \circ i : \mathbb{I}_K \to GL(V_\ell)$ is trivial on each $U_{v,m}$ when $\mathfrak{p}_v \neq \ell$, and
- $\rho \circ i(x) = f(x^{-1})$ for $x \in \prod_{v|\ell} U_{v,m} \subseteq T_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$.

Idea: "m captures the ramification of ρ".

Existence of modulus of definition

Here is a general fact (cf. Serre's book page III-11)

Theorem

Any abelian ℓ-adic representation of K has only a finite number of ramified places.

From this it easily follows:

Theorem

Every locally algebraic abelian ℓ-adic representation has a modulus of definition.

PART 3: The relation with S_m

The universal property of S_m

Recall the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times/O_K^\times \cap U_m & \longrightarrow & \mathbb{I}_K/U_m & \longrightarrow & C_m & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_m(\mathbb{Q}) & \longrightarrow & S_m(\mathbb{Q}) & \longrightarrow & C_m & \longrightarrow & 1 \end{array}$$

Something we did not mention is that S_m is **universal** for the fist square: Any commutative diagram

$$\begin{array}{ccc} K^\times/O_K^\times \cap U_m & \longrightarrow & \mathbb{I}_K/U_m \\ \downarrow & & \downarrow \\ T_m(\mathbb{Q}) & \longrightarrow & A(\mathbb{Q}) \end{array}$$

with A a commutative algebraic group, factors through S_m.

The representations from S_m are locally algebraic

Reminder/Notation. K a number field, m a modulus.
Let $\phi : S_{m,\mathbb{Q}_\ell} \to GL(V_\ell)$ be a representation of algebraic groups. Using the canonical representation $\epsilon_\ell : G^{ab} \to S_m(\mathbb{Q}_\ell)$ we get the representation

$$\phi_\ell = \phi \circ \epsilon_\ell : G^{ab} \to GL(V_\ell).$$

Theorem

ϕ_ℓ is locally algebraic with associated morphism of algebraic groups

$$f : T_{\mathbb{Q}_\ell} \to GL(V_\ell)$$

given by $\phi \circ \pi$ where

$$\pi : T \to T_m \to S_m.$$

This follows easily from the constructions and definitions, not much to prove. However, there is a (partial) converse which is much more delicate.

The main theorem

Theorem (Main theorem of the minicourse)

Let $\rho : G^{ab} \to GL(V_\ell)$ be an ℓ-adic **abelian** *representation. Assume:*

- *ρ is* **rational**
- *ρ is* **locally algebraic**, *with modulus of definition m.*

Then there is a $\mathbb{Q}$-vector space V_0 with a morphism $\phi_0 : S_m \to GL(V_0)$ of algebraic groups over $\mathbb{Q}$, such that $\rho = \phi_\ell$, with ϕ_ℓ constructed from ϕ_0.

We recall that ϕ_ℓ is the bottom row:

$$\begin{array}{ccccc} & & S_m(\mathbb{Q}) & \xrightarrow{\phi_0} & GL(V_0) \\ & & \downarrow & & \downarrow \\ G^{ab} & \xrightarrow{\epsilon_\ell} & S_m(\mathbb{Q}_\ell) & \xrightarrow{\phi_{0,\ell}} & GL(V_{0,\ell}) \end{array}$$

Proof

Let $f : T_{\mathbb{Q}_\ell} \to GL(V_\ell)$ be the algebraic map associated to the locally algebraic abelian representation ρ. Since m is a modulus of definition, we have

$$\rho \circ i(x) = f(x^{-1}) \quad \forall x \in K_\ell^\times \cap U_m = \prod_{v|\ell} U_{v,m}.$$

(The next computation explains the exponent -1)
Define $\psi : \mathbb{I}_K \to GL(V_\ell)$ by

$$\psi(x) = (\rho \circ i(x)) \cdot f(x_\ell), \quad x_\ell \in K_\ell^\times \text{ the } \ell\text{-component of } x \in \mathbb{I}_K.$$

Remark: ψ is trivial on U_m and it equals f on $K^\times \subseteq \mathbb{I}_K$ (the Artin map i kills $K^\times \subseteq \mathbb{I}_K$).
Consequence: $\boxed{f \text{ is trivial on } K^\times \cap U_m.}$
Hence, $f : T_{\mathbb{Q}_\ell} \to GL(V_\ell)$ factors through $f_m : T_{m,\mathbb{Q}_\ell} \to GL(V_\ell)$

Proof

Using the universal property of S_m one checks (small computation) that for $f_m : T_{m,\mathbb{Q}_\ell} \to GL(V_\ell)$ there is a morphism of group schemes over $\mathbb{Q}_\ell$

$$\phi : S_{m,\mathbb{Q}_\ell} \to GL(V_\ell)$$

such that the following holds:

(i) f_m factors through ϕ: that is $T_{m,\mathbb{Q}_\ell} \to S_{m,\mathbb{Q}_\ell} \to_\phi GL(V_\ell)$ where the first arrow is the canonical one.

(ii) The map $\mathbb{I}_K \to_\epsilon S_m(\mathbb{Q}_\ell) \to_\phi GL(V_\ell)$ is ψ.

For a computation below, we recall that

$$\alpha_\ell : \mathbb{I}_K \to \prod_{v|\ell} K_v^\times = (K \otimes \mathbb{Q}_\ell)^\times = T(\mathbb{Q}_\ell) \to T_m(\mathbb{Q}_\ell) \to S_m(\mathbb{Q}_\ell).$$

where the last arrow is that of item (i).

Proof

Claim. $\phi_\ell = \rho$. Indeed

$$\begin{aligned}\phi_\ell \circ i(x) &= \phi(\epsilon_\ell(x)) = \phi(\epsilon(x) \cdot \alpha_\ell(x)^{-1}) \\ &= \psi(x) \cdot \phi(\alpha_\ell(x))^{-1} \\ &= \rho(i(x)) f(x_\ell) \phi(\alpha_\ell(x))^{-1} = \rho(i(x))\end{aligned}$$

because $f(x_\ell) = \phi(\alpha_\ell(x))$ by the last remark in the previous slide. This proves the claim.

It remains to show that $\phi_\ell = \rho$ comes from certain $\phi_0 : S_m \to GL(V_0)$ defined over $\mathbb{Q}$. But this is (once again!) by the test of Frobenius elements, and ϕ_ℓ descends to $\mathbb{Q}$ because $\rho = \phi_\ell$ is rational. $\square$

Main corollary

Corollary (Main consequence)

Let $\rho : G^{ab} \to GL(V_\ell)$ be an ℓ-adic abelian representation. Assume:

- *ρ is rational*
- *ρ is locally algebraic, with modulus of definition m.*

For each ℓ' there is a unique ℓ'-adic rational rep. $\rho_{\ell'} : G^{ab} \to GL(V_{\ell'})$ compatible with ρ. It is **semisimple** *and* **locally algebraic***.*
These $\rho_{\ell'}$ from a strictly compatible system with ramification in m.
For an infinite number of primes ℓ', the ℓ'-adic representation $\rho_{\ell'} : G^{ab} \to GL(V_{\ell'})$ is diagonalizable over $\mathbb{Q}_{\ell'}$ (it splits as sum of 1-dimensional representations of G^{ab}.)

Remark. 1-dimensional rep's of G^{ab} are also called **Hecke characters**.

This is the main output of the minicourse, together with the Hodge–Tate test for being locally algebraic.

End of day 3 (and the minicourse)