Diophantine Approximation and Diophantine Equations
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Consider the three following statements.

(TM) *Thue-Mahler equation.*

For any algebraic number field K, any finite set S of places of K containing all the archimedean places, any $k \in K^\times$ and any binary homogeneous form $F(X,Y)$ having at least three non proportional linear factors in $\mathbb{C}$, the equation

$$F(x,y) = k\epsilon$$

has but a finite number of classes of solutions $(x,y,\varepsilon) \in \mathcal{O}_S^2 \times \mathcal{O}_S^\times$.

Recall that two solutions (x,y,ε) and (x',y',ε') in $\mathcal{O}_S^2 \times \mathcal{O}_S^\times$ of the equation $F(x,y) = k\epsilon$ are equivalent if there exists a unit u in $\mathcal{O}_S^\times$ such that $x' = ux$, $y' = uy$, $\epsilon' = u^d\varepsilon$, where d is the degree of F.

(S) *Siegel S-unit equation.*

For any algebraic number field K and any finite set S of places of K containing all the archimedean places, the equation

$$\epsilon_1 + \epsilon_2 = 1$$

has but a finite number of solutions $(\varepsilon_1, \varepsilon_2)$ in $\mathcal{O}_S^\times \times \mathcal{O}_S^\times$.

(HE) *Hyperelliptic equation.*

For any algebraic number field K, any finite set S of places of K containing all the archimedean places and any polynomial f in $K[X]$ with at least three distinct complex roots, the orders of multiplicity of which are odd, the equation

$$y^2 = f(x)$$

has but a finite number of solutions (x,y) in $\mathcal{O}_S \times \mathcal{O}_S$.

(SE) *Superelliptic equation.*

For any algebraic number field K, any finite set S of places of K containing all the archimedean places, any integer $m \geq 3$ and any polynomial f in $K[X]$ with at least two distinct complex roots , the orders of multiplicity of which are prime to m, the equation

$$y^m = f(x)$$

has but a finite number of solutions (x,y) in $\mathcal{O}_S \times \mathcal{O}_S$.

In the course of January 9, 2025, a sketch of proof was given for each of the implications

$$(S) \Longrightarrow (TM), \quad (S) \Longrightarrow (HE), \quad (TM) \Longrightarrow (SE).$$

The exercise is to select one (or two, or three) of these implications and to give a complete proof filling all the details.