F field.
Recall the following basic fact from Module 1:

• Let $f \in \mathbf{F}[x]$ be an irreducible polynomial. Then there exists a field extension $\mathbf{F} \subset \mathbf{K}$ with the property that f has a zero in $\mathbf{K}$.
• $\mathbf{K} \cong \mathbf{F}[x]/(f)$.

The following proposition was stated and illustrated.

Proposition. (Milne, Prop.2.4) *For every polynomial* $f \in \mathbf{F}[x]$*, there exists a field extension* $\mathbf{F} \subset \mathbf{K}$ *with the property that*
(a) f*, as a polynomial in* $\mathbf{K}[x]$*, decomposes as* $f(x) = c(x - \alpha_1) \ldots (x - \alpha_m)$*, with* $\alpha_i \in \mathbf{K}, c \in \mathbf{K}$;
(b) $\mathbf{F}(\alpha_1, \ldots, \alpha_m) = \mathbf{K}$.

In addition,
(c) the field $\mathbf{K}$ *is unique up to* $\mathbf{F}$*-isomorphism;*
(d) If $\deg(f) = n$*, then* $[\mathbf{K} : \mathbf{F}] \leq n!$.

By definition, $\mathbf{K}$ *is a splitting field of* f *over* $\mathbf{F}$.
Note that it depends both on f *and on* $\mathbf{F}$.

Examples:
Denote by $\mathbf{F}_f$ a splitting field of f over $\mathbf{F}$.
$f = X^2 - 2$, $\mathbf{Q}_f = \mathbf{Q}(\sqrt{2})$;
$f = X^2 - 2$, $\mathbf{R}_f = \mathbf{R}$;
$f = X^2 + 1$, $\mathbf{Q}_f = \mathbf{Q}(i)$;
$f = X^2 + 1$, $\mathbf{R}_f = \mathbf{R}(i) = \mathbf{C}$;
$f = X^3 - 1$, $\mathbf{Q}_f = \mathbf{Q}(\omega)$, where $\omega = e^{i2\pi/3}$;
$f = X^3 - 1$, $\mathbf{R}_f = \mathbf{R}(\omega) = \mathbf{C}$;
$f = X^3 - 2$, $\mathbf{Q}_f = \mathbf{Q}(\sqrt[3]{2}, \omega)$.

Excercise. *Consider the real number* $\alpha = \sqrt{2} + \sqrt[3]{2}$*. Determine the minimal polynomial of* α *in* $\mathbf{Q}[x]$.