NAP 2019, CLASS #6, MAY 15, 2019

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Notes: 1. Please read sections 3.6, 3.7 of the book before class Thursday and sections 3.8 and Chapter 5 of the book before Friday.

2. We are sorry we arrived late to class Wednesday after our hiking from Daman to Markhu. We were slow at the hiking and then had many difficulties with construction, traffic and blockage on the roads. We were afraid we would never get back! We were pleased that we could still teach the class, because you waited for us to arrive. It was great that you were discussing the material with each other when we came.

• In section 3.5 on PIDs (Principal ideal domains), we proved the Division Algorithm for polynomial rings over fields:

Theorem 3.7: The polynomial ring $K[x]$ satisfies the Division Algorithm, if K is a field. That is, for every pair $f(x), g(x) \in K[x]$, where $f(x) \neq 0$, there exist $q(x), r(x) \in K[x]$ such that

$$g(x) = \boxed{q(x)} f(x) + \boxed{r(x)}, \quad \text{where } r(x) = 0 \quad \text{or } \deg r(x) < \deg f(x).$$

Illustrated the proof by showing long division of $2x^5 + 3x^3 + 1$ by $3x^2 + 2$, or something similar.

• Proved

Theorem 3.8: The polynomial ring $K[x]$ is a PID, if K is a field.
(Used Division Algorithm.)

• Introduced a "common sense" lemma we use:

2/3 **Lemma:** ("two out of three lemma": If something works for 2 of the 3 parts of a simple equation, then it works for the third.)

(1) (Version for $\mathbb{Z}$) If a, b, c and $d \in \mathbb{Z}$, $d \neq 0$ and

$$a + b = c \quad \text{or } a - b = c,$$

then d divides two of $\{a, b, c\} \implies d$ divides the third.

(2) (Version for ideals of a ring R) If $a, b, c \in R$ and I is an ideal of R, and

$$a + b = c \quad \text{or } a - b = c,$$

then two of $\{a, b, c\} \in I \implies$ the third is in I.

The 2/3 Lemma comes up all the time, and we'll often use it without comment. But you should know how to prove it. (Hint: Divide into cases. Do one case thoroughly, then say "Similarly" for the others.)

• Proved

Theorem 3.9: Every principal ideal domain is a unique factorization domain.

We followed Garling's proof pretty closely. We pointed out that in any ring the union of an ascending chain $I_1 \subseteq I_2 \subseteq IU_3 \subseteq \dots$ is always an ideal. The key idea is "Everybody's gotta be somewhere!", that is, an element of the union has to be in some I_n.