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• In Section 3.7, we use an easy consequence of Theorem 3.11, namely

Fact 3.11$^+$: Let R be a UFD with field of fractions F. If $g, g' \in R[x]$ are primitive polynomials and $\beta \in F$ is such that $g = \beta g'$, then $\beta \in R$ and β is a unit of R.

• **Corollary (Gauss' lemma):** Let R be a UFD with field of fractions F. Let $g \in R[x]$, $g \neq 0$. Then g is irreducible in $R[x] \iff$ either (i) or (ii) holds:

(i) $\deg g = 0$ and g is irreducible in R, or
(ii) g is primitive and irreducible as an element of $F[x]$.

• **Theorem 3.13:** Let R be a UFD. Then $R[x]$ is a UFD.

Thus $\mathbb{Z}[x_1, \ldots, x_n]$ and $\mathbb{Q}[x_1, \ldots, x_n]$ are UFDs.

• SHOULD HAVE DONE (Please read.)

Corollary 1: Let R be a UFD. If $f, g \in R[x]$, f primitive in $R[x]$, and $f \mid g$ in $F[x]$, then $f \mid g$ in $R[x]$. That is,

$$\text{If } f, g \in R[x], f \text{ primitive in } R[x], \text{ and } \exists h \in F[x] \text{ with } fh = g, \text{ then } h \in R[x].$$

Proof: There exist elements $\beta \in F \setminus \{0\}$ and $c \in R$ such that $h = \beta h_1$, and $g = cg_1$, and h_1 and g_1 are primitive poynomials in $R[x]$. (Clear denominators, factor out the content, etc.) Now we have $f\beta h_1 = cg_1$, that is, $c^{-1}\beta f h_1 = g_1$. Now fh_1 and g_1 are primitive, so by **Fact 3.11$^+$**, $c^{-1}\beta$ is a unit of R, say $c^{-1}\beta = u \in R$. Now we have $\beta = uc \in R$, so $h = \beta h_1 \in R[x]$.

• In Sections 3.8 and 3.9, R is a general commutative ring. Defined *prime ideal*, *maximal ideal*. Proved

Proposition: Every maximal ideal is prime.

• Discussed **Correspondence Theorem for quotient rings:** If J is an ideal of a ring R and $\pi : R \to R/J$ is the coset map, then

$$\{ \text{ ideals } I \text{ of } R \mid J \subseteq I \subseteq R\} \text{ and } \{ \text{ ideals } \overline{I} \text{ of } R/J\}$$

are in one-to-one correspondence, via $I \to \pi(I) = I/J$, for every ideal I of R containing J and $\overline{I} \to \pi^{-1}(\overline{I})$, for every ideal $\overline{I}$ of R/J.

• Discussed, partly proved

Theorem 3.15: A ring R is a field $\iff$ $\{0\}$ and R are the only ideals of R.

Theorem 3.16: If J is a proper ideal of a ring R, then the ring R/J is a field $\iff$ J is a maximal ideal of R.

• Proved three irreducibility tests related to Chapter 5:

Theorem: Let $f \in F[x]$, where F is a field, and let $a \in F$. Then a is a root of f (that is, $f(a) = 0$) $\iff$ $(x - a) \mid f$.

Theorem: Let $f = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$, where R is a UFD with field of fractions F. Then every root b/c of f in F, where $b, c \in R$ are in lowest terms, satisfies $b \mid a_0$ and $c \mid a_n$ in R.

Theorem 5.2 (Eisenstein's criterion: Let R be an integral domain, and let $f = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$, where no nonunit of R divides every a_i. Suppose that p is a prime element of R such that: $p \mid a_i, \forall i$ with $0 \leq i < n$, $\quad p \nmid a_n$, $\quad$ and $p^2 \nmid a_0$. Then f is irreducible in $F[x]$.