NAP 2019 - MODULE V - CLASS #5
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- We solved exercises 12.1 and 12.11 in Garling's book.

- Assume $char(K) \neq 2$. For a polynomial $f \in K[X]$, with a splitting field extension L/K, we defined the *discriminant* Δ, and showed that
  - $\Delta \neq 0$ if and only if f is separable.
  - if f is separable then $\mathrm{Gal}(L/K) \subseteq A_n$ if and only if Δ as a square root in K.

- We showed that the discriminant can be computed as a determinant of a symmetric $n \times n$ matrix with coefficients in K, which are symmetric polynomials in the roots of f and thus polynomials in the coefficients of $f(X)$.

- in particular for a cubic polynomial $f(X) = X^3 + aX^2 + bX + c$ we get

$$\Delta = a^2b^2 + 18abc - 4b^3 - 4a^3c - 27c^2;$$

- By a change of variables, every cubic polynomial can be transformed to a polynomial of the form $f(X) = X^3 + pX + q$. Then $\Delta = -4p^3 - 27q^2$ (there is a misprint on Garling's book!).

- Let $f(X)$ be an irreducible cubic polynomial in $K[X]$ ($char(K) \neq 2, 3$), L/K be a splitting field extension for $f(X)$, $G = \mathrm{Gal}(L/K)$. Then $[L : K]$ can be 3 or 6, so that G can be either A_3 or S_3. More precisely we have the following:

Theorem

- If Δ has a square root in K then $[L : K] = 3$ and $G = A_3$.
- If Δ has not a square root in K then $[L : K] = 6$ and $G = S_3$.

In both case we can solve f by radicals on K if and only if we can solve it on $K(\sqrt{\Delta})$.